1980

Aspects of symmetric Bayesian sequential decision problems

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ASPECTS OF SYMMETRIC BAYESIAN SEQUENTIAL DECISION PROBLEMS

Iowa State University PH.D. 1980

Aspects of symmetric Bayesian sequential decision problems

by

Anuchit Lamyordmakpol

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1980

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I. INTRODUCTION AND REVIEW OF LITERATURE

Some attention has been paid in the past thirty or so years to the problem of sequentially choosing from among several possible sources of information. Generally, work in this area can be categorized in two ways: first, by whether the work is concerned with non-asymptotic Bayesian considerations or asymptotic non-Bayesian ones; second, by whether the criterion in accordance with which procedures are judged is of "decision-theoretic" type, with decision and sampling losses, or of "two-armed-bandit" type, where it is a matter of the magnitudes of generated sums. We thus can speak in short hand of "Bayesian decision-theoretic", "Bayesian two-armed-bandit", "asymptotic decision-theoretic", and "asymptotic two-armed-bandit" problems, which, for ease of reference, we shall simply denote below as being of the first, second, third and fourth type. This thesis is concerned with problems of the first type.

The earliest work in this general area was that of Robbins (28), which was of the fourth type. This was followed by Isbell (17), by Vogel (32,33), and by Robbins and Siegmund (29). The latter paper actually has elements of the third type, since the criterion there is taken to be not only that the source generating small values be sampled often, but also that the decision which of the two sources does generate small values be made in an efficient way. A similar theme, except pertaining to variances rather than means, is taken up by Werner (38).

Work wholly devoted to the third type of problem was initiated by Chernoff (8), and was extended, among others, by Albert (1) and Kiefer and Sacks (18).

The papers of Bradt, Johnson and Karlin (6) on the one hand, and of Bradt and Karlin (7) and also Borwanker, David and Ingwell (4) on the other hand, are respectively of second and first type. All three have as their main concern the following question: Under what conditions is a "myopic Bayesian", (that is, a Bayesian acting as if his current action were his last), an "optimal Bayesian". A version of this, in its type-one setting, is in fact one of the issues addressed below. This and other areas treated in this thesis are outlined below. Generally the method of attack has been to extend and adapt methodology developed in (3) for the one-source case, to fit the two-source problems of interest to us.

In Chapter II we present a general structure for "type-F" problems which involve a finite number of states of nature, a finite number of actions, and a finite number of possible sources, with possibly different costs. Additional ingredients required in the framework of "sequential design of experiments" are added, such as a measure of belief in terms of a "prior probability distribution" over the state-of-nature space, the likelihoods corresponding to types of sources and states of nature, and also the loss function. Since we are going to be Bayesians, it is required to deal with prior and posterior probability distributions; hence, formulations for computing the posterior probability distributions corresponding to a single observation or a set of observations also are given. We then define a sequential decision policy δ in terms of a sequence of decision criteria δ_t for both "bounded" and "unrestricted" sequential decision problems, where t is decision time. A particular class of policies δ, namely "Bayes-like" sequential decision policies, is mentioned. Reasons for considering only this class throughout this thesis are given. We also categorize the above class into two subclasses: "non-stationary" and "stationary" Bayes-like classes, the former being important for bounded sequential decision problems, and the latter for unrestricted problems. We note that we may think of the space of prior and posterior probability distributions for problems of type-F as a simplex in Euclidean space, and we point out in Section II.B. that a sequential decision policy may be described in terms of a sequence of partitions of that simplex. We note as well that risks of Bayes-like policies obey iterative schemes, of which the stationary ones are discussed in Section II.B., and the non-stationary ones in Section III.A.

Much of the discussion in Chapter III concerns the issue of problem truncation in the case of one source of information. In Section III.A., the introduction of the chapter, the concepts of "effectively N-truncated" and "effectively non-truncated" are discussed. The main concerns of Section III.B. are to explore conditions for a problem to be effectively non-truncated. Theorem III.1 gives the main result of the section, pertaining to the continuous case. This theorem involves the notion of "generation" of a posterior probability distribution by a prior probability distribution, and the requirement that there exist a particular prior probability distribution such that the set of posteriors generatable by that prior in fact comprises the set of all possible priors. This actually is the only place in this dissertation where the fact that the set of all evolving posterior may not in fact comprise the set of all possible priors is of importance. A modification of the conditions of Theorem III.1 is needed to cover the discrete case. An example is given involving the normal case. Conditions for a problem to be effectively truncated are discussed in Section III.C; Lemma III.3 concerns 0-truncation, and Lemma III.4 concerns the specialized 0-L terminal decision loss. A binomial example is discussed in detail, in the context of effective 0-truncation and effective 1-truncation (cf. Lemma III.5).

Chapter IV is devoted to a discussion of the concepts of source selection and source expendability in the case of two sources of information and two states of nature. A general set-up for problems under consideration is presented in Section IV.A. We lean here on the prior result that the Bayes risk $V^2(\xi)$ of the optimal Bayes policy of unrestricted sequential decision problems with two sources of information satisfies the functional equation:

$$V^2(\xi) \quad = \quad \min\{V_o^2(\xi),\ V_x^2(\xi),\ V_y^2(\xi)\} \qquad ,$$

where $V_o^2(\xi)$ is the optimal Bayes risk of the no-data policy, $V_x^2(\xi)$ is the optimal Bayes risk for taking X first and then pursuing the optimal policy corresponding to $V^2(\xi)$ thereafter, and similarly for $V_y^2(\xi)$. Then the definitions of source selection and source expendability are given in terms of comparisons between the magnitudes of $V_x^2(\xi)$ and $V_y^2(\xi)$ for those ξ such that $V_o^2(\xi) \geq \min\{V_x^2(\xi),\ V_y^2(\xi)\}$. Section IV.B. is reserved for a development of distribution functions of the likelihood ratios for both source X and source Y. In Section IV.C certain conditions are imposed on these distribution functions so that each source gives us the same amount of information for deciding which state of nature obtains, making the source with higher sampling cost expendable. This section also contains more detailed information on source expendability for the binomial case. Section IV.D. contains the extension to the sequential case of the result of Bradt and Karlin mentioned earlier. Chapter IV closes with Section IV.E., devoted to illustrating by example that sampling cost is less decisive, compared to wrong-decision loss, in the short run than in the long run. The discussion here is related to the prior work of third type.

Chapter V is devoted to a discussion of symmetry. It contains conditions insuring symmetry of the risk function in special cases, as well as two examples.

II. GENERAL FEATURES OF BAYESIAN SEQUENTIAL DECISION PROBLEMS

A. Prior and Posterior Probabilities

In general, sequential decision theory (3,8,16,23,27,28,36,40) has to do with what actions to take and/or when and how to take data in the face of uncertainty; for instance, uncertainty about a population parameter of interest which may have one of a number of possible values. Being Bayesian, we would admit a measure of belief about such a parameter, in terms of a "prior probability distribution". There is also typically uncertainty regarding the values of data to be generated, measured in terms of likelihood w.r.t. (with respect to) particular population parameter values. A variety of problems involve as well the choices of data attainable from one of a number of possible sources. Decision models of this type may be structured with a finite number of actions, pertaining both to whether and how to sample and what terminal decision to take, a finite number of population parameters (or states of nature), a finite number of independent sources of information (or experiments) with possibly different sampling costs, and decision losses depending on whether or not one has made the right decision. The statistician's options are typically described by sequential decision policies δ (36) which denote a sequence of decision criteria $\{\delta_t\}$, where δ_t is a rule for choosing actions at decision period $t = 0,1,2, \ldots$. We shall designate δ by δ^∞ when the sequence $\{\delta_t\}$ is infinite and by δ^N when the sequence $\{\delta_t\}$ is finite; that is,

$$\delta^\infty = \{\delta_t^\infty\}_{t=0}^{\infty} \quad ,$$

$$\delta^N = \{\delta_t^N\}_{t=0}^N ,$$

where the superscripts of both sequences carry the information about whether we are dealing with a "non-truncated" or "truncated" decision problems. A statistician's computations regarding a sequential decision problem simply concern taking a particular δ and computing the expected loss under δ, where the expectation is w.r.t. the prior probability distribution and the likelihoods. Our general desire is to look for a policy δ_Δ^*, "the Bayes policy", which has the smallest Bayes risk within a class of policies Δ. Throughout this dissertation, it is of interest to consider only Δ's that are classes of "Bayes-like" policies δ whose criteria δ_t are functions of the "current" prior or posterior probability distribution. The reasons for this are as follow: (1) for the "truncated" situations of Chapter III, it is readily verified that there is always a Bayes-like Bayes policy; (2) there is no problem known to the author for which there is not a Bayes-like Bayes policy; (3) it will be evident that, for the special case of two states of nature and one information source, our Bayes-like policies are just "generalized sequential probability ratio tests" (19) and these form a complete class in that special case.

As indicated above, we shall specialize in this dissertation to sequential decision problems of type F; that is, sequential decision problems involving:

(a) a finite space of actions: $A = \{a_1, a_2, \ldots, a_l\}$, (2.1)

(b) a finite space of states of nature: $H = \{\theta_1, \theta_2, \ldots, \theta_m\}$, (2.2)

and

(c) a finite space of experiments: $E = \{e_1, e_2, \ldots, e_n\}$. (2.3)

The elements of H and E determine stochastic processes over a sample space of elements $s \in S$, which, for our purposes, will be specialized to an infinite collection of independent random variables with probability densities (w.r.t. a suitable σ-finite jointly dominating measure μ)

$$f(s_i | e_i, \theta_j) = f_{ij}(s_i) \tag{2.4}$$

on S x E x H, where s is made to carry the subscript of the experiment that generates it.

Additional ingredients of Bayesian sequential decision problems are:

(a) a loss $L(a,\theta)$ defined on A x H, which may be sampling cost or decision loss,

(b) a prior probability distribution: $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$, (2.5)

where

(c) $\xi_j = P_r\{\theta_j$ is the true state of nature$\} \geq 0$, (2.6)

and

$$\sum_{j=1}^{m} \xi_j = 1 \quad . \tag{2.7}$$

Suppose that $\underline{\xi}$ is a given prior probability distribution over H and s_i is an output of source of information e_i; then the prior probability distribution determines a posterior probability distribution

$$T(\underline{\xi}, s_i) = (T_1(\underline{\xi}, s_i), T_2(\underline{\xi}, s_i), \ldots, T_m(\underline{\xi}, s_i)) \quad , \tag{2.8}$$

where

$$T_j(\underline{\xi}, s_i) = \frac{\xi_j f_{ij}(s_i)}{\sum_{j=1}^{m} \xi_j f_{ij}(s_i)} . \tag{2.9}$$

It is equally true that several outputs from several possibly different sources can be used to update a prior probability distribution to a current posterior probability distribution. To be specific, let $s^r = \{s_{i(1)}, s_{i(2)}, \ldots, s_{i(r)}\}$ be a set of r outcomes, with subscripts indicating the respective experiments performed; then the posterior probability distribution is given by

$$T(\underline{\xi}, s^r) = (T_1(\underline{\xi}, s^r), T_2(\underline{\xi}, s^r), \ldots, T_m(\underline{\xi}, s^r)) , \tag{2.10}$$

where

$$T_j(\underline{\xi}, s^r) = \frac{\xi_j \prod_{k=1}^{r} f_{i(k)j}(s_{i(k)})}{\sum_{j=1}^{m} \xi_j \prod_{k=1}^{r} f_{i(k)j}(s_{i(k)})} . \tag{2.11}$$

B. Bayes-Like Policies

We have already specified in Section II.A. certain basic components of the Bayesian sequential decision problems to be discussed in this thesis. The present section is devoted to a more detailed discussion of those components and also to the introduction of certain classes of decision policies such as Bayes-like, stationary and truncated. The development of the following formulations is aimed at the discussion of specific problems in later chapters.

As we have already mentioned in Section II.A., our sequential decision problems are concerned with policies δ which call for either sampling one of the available sources of information of stopping and taking a terminal decision. We find it useful in this connection to partition the finite space of actions A (cf. (2.1)) into two subspaces of actions; that is,

$$A = \{A_E, A_T\} \quad ,$$

where

$$A_E = \{a_{e_1}, a_{e_2}, \ldots, a_{e_n}\} \quad ,$$

$$A_T = \{a_1, a_2, \ldots, a_m\} \quad , \quad m + n = \ell \quad ,$$

a_{e_i} = action calling for taking one observation from source of information e_i; $i = 1,2, \ldots, n$,

and

a_j = action calling for stopping taking any further observation, and accepting θ_j; $j = 1,2, \ldots m$.

Note that the loss $L(a,\theta)$ in Section II.A. is now specialized to sampling cost if $a \in A_E$, and to decision loss if $a \in A_T$, as follows:

$$L(a_{e_i}, \theta_j) = c_i \quad (>0) \quad \text{for } j = 1,2, \ldots, m \quad . \tag{2.12}$$

$$L(a_j, \theta_k) = L \quad (>0) \quad \text{for } j \neq k \tag{2.13}$$

$$= 0 \quad \text{for } j = k;\ j,k = 1,2, \ldots, m \quad . \tag{2.14}$$

That is, $L(a_{e_i}, \theta_j)$ is a sampling cost per experiment e_i and $L(a_j, \theta_k)$ is a decision loss due to accepting θ_j when θ_k is in fact the true state of nature.

Definition II.1

A decision loss function defined on $A_T \times H$ is of the form 0 - L if $L(a_j, \theta_R)$ satisfies both (2.13) and (2.14).

In the sequential decision problem we must make successive decisions. At decision period t = 0,1,2, ..., an action in A must be chosen by δ_t. When we follow a policy δ an action $a_j \varepsilon A_T$ either is chosen, so that the decision process is terminated and we incur a decision loss $L(a_j, \theta_k)$, or, on the other hand, an action $a_{e_i} \varepsilon A_E$ is chosen and we must pay a sampling cost c_i and observe the data value s_i. The criteria δ_t in general will depend on the initial prior distribution and the entire sample history (by which we mean the total collection of data already obtained thus far) and may in fact depend on these only through the current posterior probability distribution and t. As indicated in Section II.A., it is natural for us to restrict ourselves to δ's composed of the latter type of δ_t (henceforth called "Bayes-like" δ's), and then to look for a good policy within this class.

Definition II.2

A policy $\delta = \{\delta_t\}$ is said to be Bayes-like if its criteria δ_t are non-randomized and only depend on the current posterior probability distribution and decision period t.

Definition II.3

δ is stationary Bayes-like if it is Bayes-like and δ_t is constant w.r.t. t.

We note that the sequential decision problem discussed in this section can be considered as a Markov process with absorbing states, where absorption corresponds to the sequential decision process terminating when δ calls for an action in A_T.

Let

$$\Xi = \{\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_m): \ \xi_j \geq 0, \ \sum_{j=1}^{m} \xi_j = 1\} \tag{2.15}$$

be a space of prior probability distribution over $H = \{\theta_1, \theta_2, \ldots, \theta_m\}$, and $\delta = \{\delta_t\}$ be a Bayes-like policy; then, for $t = 0,1,2, \ldots,$ define:

$$\Xi_{e_i}^{\delta_t} = \{\underline{\xi} \in \Xi: \ \delta_t(\underline{\xi}) = a_{e_i}\}; \ i = 1,2, \ldots, n \quad , \tag{2.16}$$

and

$$\Xi_{j}^{\delta_t} = \{\underline{\xi} \in \Xi: \ \delta_t(\underline{\xi}) = a_j\}; \ j = 1,2, \ldots, m \quad . \tag{2.17}$$

We now consider the description of a Bayes-like policy δ, denoted by $D(\delta)$, in terms of the continuation region $\Xi_{e_i}^{\delta_t}$ and stopping regions $\Xi_{j}^{\delta_t}$; that is,

$$D(\delta) = \{\Xi_{e_1}^{\delta_t}, \Xi_{e_2}^{\delta_t}, \ldots, \Xi_{e_n}^{\delta_t}; \Xi_{1}^{\delta_t}, \ldots, \Xi_{m}^{\delta_t}\} \quad . \tag{2.18}$$

For convenience, we shall denote the R.H.S. (right hand side) of (2.18) by $\{\{\Xi^{\delta}_{e_i}{}^{t}\};\ \{\Xi^{\delta}_{j}{}^{t}\}\}$ and write

$$D(\delta) = \{\{\Xi^{\delta}_{e_i}{}^{t}\};\ \{\Xi^{\delta}_{j}{}^{t}\}\} \tag{2.19}$$

with the t sub-superscript and outside brackets removed when δ is stationary. The Bayes risk of a stationary Bayes-like δ, denoted by $R_\delta(\underline{\xi})$, with respect to the prior $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$ and the description $\{\{\Xi^{\delta}_{e_i}\};\ \{\Xi^{\delta}_{j}\}\}$, is given by (10)

$$R_\delta(\underline{\xi}) = L \sum_{j=1}^{m} (1 - \xi_j)\, 1_{\{\Xi^{\delta}_{j}\}} + \sum_{i=1}^{n} [c_i + \int_S R_\delta\,(T(\underline{\xi}, s_i))\, \bar{f}_i^{\,\underline{\xi}}\,(s_i)d\mu\,(s_i)]\, 1_{\{\Xi^{\delta}_{e_i}\}} \quad , \tag{2.20}$$

where $1_{\{.\}}$ is the indicator function of the subset $\{.\}$, μ is the dominating measure w.r.t. which the f_{ij}'s are densities,

$T(\underline{\xi}, s_i)$ is as defined in (2.8) and (2.9),

and

$$\bar{f}_i^{\,\underline{\xi}}\,(s_i) = \sum_{j=1}^{m} \xi_j\, f_{ij}\,(s_i) \quad . \tag{2.21}$$

III. TRUNCATION IN THE CASE OF ONE SOURCE OF INFORMATION

A. Truncated and Non-truncated Problems

In any practical situation there are certain factors, such as a deadline or a budget, which play important roles in limiting ourselves to considering a particular class of policies calling for bounded sample size. This class is commonly called the class of "truncated" policies.

Consider a Bayesian sequential decision problem of type F, as defined in Section II.A., and a Bayes-like policy $\delta = \{\delta_t\}$ with description $D(\delta) = \{\{\Xi_{e_i}^{\delta_t}\}; \{\Xi_j^{\delta_t}\}\}$, where the $\Xi_j^{\delta_t}$'s are as defined in (2.16) and (2.17). For our purposes, we require δ_t to be defined on each point $\underline{\xi}$ of Ξ; that is,

$$\Xi = \left(\bigcup_{i=1}^{n} \Xi_{e_i}^{\delta_t}\right) \cup \left(\bigcup_{j=1}^{m} \Xi_j^{\delta_t}\right), \ \forall t \quad . \tag{3.1}$$

<u>Definition III.1</u>

A policy $\delta = \{\delta_t\}$ is said to be N-truncated, denoted by δ^N, if there is an M such that $\bigcup_{i=1}^{n} \Xi_{e_i}^{\delta_M}$ is empty, and N is the smallest such M.

It is clear that, if a policy is N-truncated, then the decision criteria δ_n, $n > N$, carry no meaning; hence an N-truncated policy admits the abbreviated structure: $\{\delta_t^N\}_{t=0}^N$ alluded to in Section II.A.

<u>Definition III.2</u>

A policy $\delta = \{\delta_t\}$ is said to be non-truncated, denoted by

$$\delta^\infty = \{\delta_t^\infty\}_{t=0}^\infty, \text{ if it is not N-truncated for any N.}$$

We note that we shall sometimes write $\delta = \{\delta_t\}$ with no superscript and no specific domain of t to mean that δ can be either non-truncated or truncated.

Definition III.3

Let $N \geq M$ be two non-negative integers; the policy δ^N is said to be a cover of δ^M if, for any $0 \leq n \leq N$, $0 \leq m \leq M$ such that $N - n = M - m$, $\delta_n^N(\underline{\xi})$ assigns the same action in A as $\delta_m^M(\underline{\xi})$, $\forall \underline{\xi}$.

In view of (2.19), it is equivalent to say that δ^N is a cover of δ^M, $N \geq M$, if the "tail sequence" of the description, consisting of $M + 1$ partition of $\underline{\Xi}$, corresponding to δ^N is identical to the entire sequence describing δ^M.

Definition III.4

A sequence of truncated policies $\{\delta^k\}_{k=0}^{\infty}$ is said to be ordered if δ^N is a cover of δ^M for all N, M with $N \geq M$; a corresponding definition holds for a finite collection $\{\delta^k\}_{k=0}^{N}$.

Consider an ordered sequence of truncated policies $\{\delta^k\}_{k=0}^{N}$ and let $R_\delta(\underline{\xi})$ be the Bayes risk of δ^k w.r.t. the prior $\underline{\xi}$, $k = 0,1, \ldots, N$. Then, in analogy to (2.20) and expression (2) of the proof of Theorem 9.3.1 in Reference (3) for the optimal Bayes policy in the case of one source of information, the Bayes risk of δ^k satisfies the following functional equation:

$$R_{\delta^k}(\underline{\xi}) = L \sum_{j=1}^{m} (1 - \xi_j)\, 1_{\{\underline{\Xi}^{\delta_o^k}_j\}}$$

$$+ \sum_{i=1}^{n} \left[c_i + \int_S R_{\delta^{k-1}} (T(\underline{\xi}, s_i)) \, \bar{f}_i^{\underline{\xi}} (s_i) d\mu (s_i) \right] 1_{\{\Xi^{\delta^k}_{o} = e_i\}} \quad , \tag{3.2}$$

$k = 0, 1, 2, \ldots, N$, where $1_{\{.\}}$ is the indicator function.

<u>Definition III.5</u>

The policy δ^{*N} is said to be (uniformly) optimal within the class of N-truncated Bayes-like policies Δ_N if $\delta^{*N} \in \Delta_N$ and

$$R_{\delta^{*N}}(\underline{\xi}) = \min_{\delta^N \in \Delta_N} R_{\delta^N}(\underline{\xi}) \triangleq V_N(\underline{\xi}), \ \forall \underline{\xi} \quad . \tag{3.3}$$

The above definition holds as well for the non-truncated class Δ_∞; that is, the policy δ^* is said to be (uniformly) optimal within the class of non-truncated policies Δ_∞ if $\delta^{*\infty} \in \Delta_\infty$ and

$$R_{\delta^{*\infty}}(\underline{\xi}) = \min_{\delta^\infty \in \Delta_\infty} R_{\delta^\infty}(\underline{\xi}) \triangleq V(\underline{\xi}), \ \forall \underline{\xi} \quad . \tag{3.4}$$

<u>Note III.1</u>

An algorithm for finding the optimal policy δ^{*N} is given by Blackwell and Girshick (3) for the sequential decision problem with two states of nature, one source of information with fixed sampling cost and 0 - L decision loss, and it is essentially shown (3) that δ^{*N} is then a cover of δ^{*M} for every $N \geq M$.

Note III.2

In view of Definition III.4 and Note III.1, the sequence $\{\delta^{*k}\}_{k=0}^{\infty}$ is ordered.

Definition III.6

A sequence of truncated policies $\{\delta^k\}_{k=0}^{\infty}$ is uniformly effectively N-truncated w.r.t. a given problem if the sequence is ordered and

$$R_{\delta^N}(\underline{\xi}) \leq R_{\delta^{N+J}}(\underline{\xi}) \quad , \quad \forall \underline{\xi} \text{ and } J \geq 0 \quad .$$

Note III.3

The concept of effectively N-truncated is equally meaningful whether or not the order property of $\{\delta^k\}_{k=0}^{\infty}$ pertains. However, effectively N-truncated seems an especially natural concept in the presence of order, in the first instance because of Note III.2 and secondly because order seems a way of specifying that all k-truncated policies of a given sequence refer to the same substantive problem.

In a given Bayesian sequential decision problem we may either attempt to find the optimal Bayes-like policy δ^{*N} within the class of N-truncated policies or we may attempt to find the optimal Bayes-like policy $\delta^{*\infty}$ without restriction on the total number of observations. It might turn out in some cases that, even if we are allowed an infinite number of observations, we do no better than if we had only been allowed a finite number (cf. (3)). For this reason, the notion of truncation is important in Bayesian sequential decision theory. Before defining what is meant by a Bayesian sequential problem being effectively truncated, we shall first draw on several results in (3) for the case of one source of information and arbitrary decision loss.

To begin with, $V_N(\underline{\xi})$ and $V(\underline{\xi})$ satisfy the following functional equations:

$$V_N(\underline{\xi}) = \min \{V_o(\underline{\xi}), c + E[V_{N-1}(T(\underline{\xi},X))]\} \quad , \tag{3.5}$$

$$V(\underline{\xi}) = \min \{V_o(\underline{\xi}), c + E[V(T(\underline{\xi},X))]\} \quad , \tag{3.6}$$

where $T(\underline{\xi},X)$ is as defined in (2.8) and (2.9), and both expectations are w.r.t.

$$\bar{f}_{\underline{\xi}}(X) = \sum_{j=1}^{m} \xi_j f_j(X) \quad . \tag{3.7}$$

In view of (3.3) and (3.4), we have

$$V_N(\underline{\xi}) \geq V(\underline{\xi}) \quad , \quad \forall \underline{\xi} \quad .$$

It is also true (3) that

$$\lim_{N \to \infty} V_N(\underline{\xi}) = V(\underline{\xi}) \quad , \quad \forall \underline{\xi} \quad . \tag{3.8}$$

It is straightforward that (3.5) implies that

$$V_1(\underline{\xi}) \leq V_o(\underline{\xi}) \quad , \quad \forall \underline{\xi} \quad . \tag{3.9}$$

It follows as well, by (3.5), (3.9), and finite induction, that

$$V_{N+1}(\underline{\xi}) \leq V_N(\underline{\xi}) \quad , \quad \forall \underline{\xi} \quad , \quad \forall N \quad . \tag{3.10}$$

Definition III.7

A Bayesian sequential decision problem is said to be effectively N-truncated if the corresponding sequence of the optimal policies $\{\delta^{*k}\}_{k=0}^{\infty}$ is uniformly effectively N-truncated.

Note III.4

In view of (3.10) and Definition III.7, a Bayesian sequential decision problem is effectively N-truncated if N is the smallest integer such that

$$V_N(\underline{\xi}) = V_{N+J}(\underline{\xi}) \quad , \quad \forall \underline{\xi} \text{ and } J \geq 0 \quad . \tag{3.11}$$

Definition III.8

A Bayesian sequential decision problem is said to be effectively non-truncated if, for any $N > M \geq 0$,

$$V_N(\underline{\xi}) \leq V_M(\underline{\xi}) \quad , \quad \forall \underline{\xi} \quad , \tag{3.12}$$

and

$$V_N(\underline{\xi}) < V_M(\underline{\xi}) \quad , \quad \text{for some } \underline{\xi} \quad . \tag{3.13}$$

Note III.5

A Bayesian sequential decision problem is effectively non-truncated if it is not effectively N-truncated for any N.

Note III.6

We may note that in (3) there is a somewhat more stringent definition of problem N-truncation: that no unrestricted Bayes policy requires more than N observations. It is readily verified that, if a Bayesian sequential decision problem is N-truncated in this sense, then it is effectively N-truncated.

We now specialize to the case of two states of nature, fixed sampling cost c and 0 - L decision loss, for which case we substitute $\xi \varepsilon \Xi = (0,1)$ for the vector $\underline{\xi}$. Then relations (3.5) and (3.6) become

$$V_N(\xi) \quad = \quad \min \{V_o(\xi),\ c + E[V_{N-1}(T(\xi,X))]\} \quad , \tag{3.14}$$

and

$$V(\xi) \quad = \quad \min \{V_o(\xi),\ c + E[V(T(\xi,X))]\} \quad . \tag{3.15}$$

<u>Lemma III.1</u> $V_N(\xi)$ is a concave function of ξ on Ξ, $\forall N$.

<u>Proof</u> (cf. (3)).

It further follows from the development in (3) that, for a fixed N, the monotonicity and concavity of $V_N(\xi)$ (cf. (3.10) and Lemma III.1) imply that, at any decision period $0 \leq t \leq N$, there exist $0 \leq \xi_L^{t,N} \leq \xi_U^{t,N} \leq 1$ such that

$$\Xi_1^{\delta_t^{*N}} = \left[\ \xi_U^{t,N}\ ,\ 1\right] \triangleq \Xi_1^{t,N} \quad ,$$

$$\Xi_2^{\delta_t^{*N}} = \left[\ 0,\ \xi_L^{t,N}\right] \triangleq \Xi_2^{t,N} \quad ,$$

and

$$\Xi_e^{\delta^*N} = \left[\xi_L^{t,N}, \xi_U^{t,N}\right] \triangleq \Xi_e^{t,N} \quad ,$$

where $\Xi_1^{\delta^*N}$, $\Xi_2^{\delta^*N}$ and $\Xi_e^{\delta^*N}$ are as defined in (2.16) and (2.17) (with e carrying no subscript since we are in the case of one source of information). An algorithm is also given in (3) for obtaining $\xi_L^{t,N}$ and $\xi_U^{t,N}$, t = 0,1, ..., N. It is shown there as well, referring to Lemma III.1 (concavity of $V_N(\xi)$) and relation (3.10) (monotonicity of $V_N(\xi)$) that

$$\Xi \supset \Xi_e^{0,N} \supset \Xi^{1,N} \supset \ldots\ldots \supset \Xi_e^{N,N} = \phi \quad , \tag{3.16}$$

where ϕ is the null set.

As mentioned earlier, the concept of effective truncation of a sequential decision problem naturally leads to the issue of source expendability. The rest of this chapter explores conditions on the parameters of a sequential decision problem of type F: sampling cost, decision loss, and probability distributions of the data involved, which allow conclusions concerning problem truncation.

B. A Condition for Effective Non-truncation

The following preliminary materials are useful for some of the results in this section.

<u>Definition III.9</u>

Let ν and μ be two measures defined on a measurable space (X,β). The measure ν is said to be absolutely continuous w.r.t. the measure μ, denoted by $\nu << \mu$, if $\nu(A) = 0$ for every set $A \in \beta$ for which $\mu(A) = 0$.

Definition III.10

The measure ν is equivalent to the measure μ, denoted by $\nu \sim \mu$, if $\nu << \mu$ and $\mu << \nu$.

Lemma III.2 Let a probability measure μ be defined on the line R^1. If $g(x)$ is a non-negative continuous function of $x \varepsilon R^1$ and $m << \mu$, m is a Lebesgue measure, then

$$\int_{R^1} g(x)\ d\mu(x) = 0 \tag{3.17}$$

implies that $g(x) = 0$ for all $x \varepsilon R^1$.

Proof Suppose that there existed $x^* \varepsilon R^1$ such that $g(x^*) = \varepsilon > 0$. Then $U = g^{-1}((\frac{\varepsilon}{2}, \frac{2\varepsilon}{3}))$ would be non-null, and also open, since g is continuous and the interval $(\frac{\varepsilon}{2}, \frac{2\varepsilon}{3})$ is open. But (31), a non-null open set of the line is the union of a countable non-null collection of disjoint open intervals, so that $m(U) > 0$, and therefore $\mu(U) > 0$, since $m << \mu$. Thus we would have

$$\int_{R^1} g(x)\ d\mu(x) \geq \int_U g(x)\ d\mu(x)$$

$$\geq \frac{3}{2}\mu(U) > 0 \quad ,$$

which would contradict (3.17).

We are now in position to give our condition for effective non-truncation, in Theorem III.1 below. Consider a problem of type F with one source of information, two states of nature (θ_1, θ_2), sampling cost c, $0 - L$ decision loss, and densities $f_1(x)$, $f_2(x)$ under θ_1, θ_2, respectively, with respect to a common σ-finite measure μ. Define, with $\xi = P(\theta_1)$,

$R(\xi,c)$ = Bayes risk of the optimal fixed sample Bayes policy based on a single observation of cost c.

Then (7)

$$R(\xi,c) = c + \xi L \int_{x:\ \frac{f_2(x)}{f_1(x)} > \frac{\xi}{1-\xi}} f_1(x)\, d\mu(x) + (1-\xi)L \int_{x:\ \frac{f_2(x)}{f_1(x)} \le \frac{\xi}{1-\xi}} f_2(x)\, d\mu(x) \quad . \tag{3.18}$$

For the special case under consideration, $V_o(\xi)$, as defined in (3.3), becomes

$$V_o(\xi) = \min\{\xi L, (1-\xi)L\} \quad . \tag{3.19}$$

We note as well that $R(\xi,c)$ can be written in terms of $V_o(\xi)$ as:

$$R(\xi,c) = c + \int_S V_o(T(\xi,x))\, \bar{f}_\xi(x)\, d\mu(x) \quad , \tag{3.20}$$

where

$$T(\xi,x) = \frac{\xi f_1(x)}{\xi f_1(x) + (1-\xi) f_2(x)} \tag{3.21}$$

$$\bar{f}_\xi(x) = \xi f_1(x) + (1-\xi) f_2(x) \quad . \tag{3.22}$$

In view of (3.5) and (3.20), we can also write

$$V_1(\xi) = \min\{V_o(\xi), R(\xi,c)\} \quad , \tag{3.23}$$

where $V_1(\xi)$ is as defined in (3.3).

We next require

Definition III.11

Let ξ^o, $\xi^* \in \Xi$. We say that ξ^o generates ξ^*, denoted by $\xi^o \to \xi^*$, if there exists $x^o \in S$ such that $T(\xi^o,x^o) = \xi^*$, where $T(\xi,x)$ is as defined in (3.21).

Theorem III.1

Suppose, for a Bayesian sequential decision problem of type F with two states of nature and one information source,

1. $m \sim \mu_i$, where m is Lebesgue measure and μ_i the probability measure corresponding to f_i, $i = 1,2$.

2. $\Xi_e^{0,1}$ contains an interval I, $I \subset \Xi$.

3. There exists $\xi^o \in \Xi_e^{0,1}$ such that ξ^o generates the open interval (0,1); that is,

$$(0,1) = \{\xi = T(\xi^o,x): \; x \in S\} \quad ,$$

where $T(\xi,x)$ is as defined in (3.21).

4. $T(\xi^o,x)$ is continuous w.r.t. x.

Then that problem is effectively non-truncated.

Proof The method of proof is to show that, if the problem is not effectively N-truncated, then the problem is not effectively (N+1)-truncated, $N \geq 0$. No more is needed since condition 2 of this theorem ensures that the problem is not effectively 0-truncated.

Suppose that the problem is not effectively N-truncated; then, by definition, there exist $\xi^*(\neq 0,1)$ such that

$$V_{N+1}(\xi^*) \quad < \quad V_N(\xi^*) \quad . \tag{3.24}$$

Take the ξ^o of condition 3 that generates ξ^*. Since $V_N(\xi)$ is monotone non-increasing w.r.t. N (cf. (3.10)), it follows that $\Xi_e^{0,N}$ is nested w.r.t. N in the sense that $\Xi_e^{0,M} \subset \Xi_e^{0,N}$ for all $M \leq N$ (cf. (3.16) and Note III.1). Since $\xi^o \in \Xi_e^{0,1}$ by condition 3, it follows that $\xi^o \in \Xi_e^{0,N+1} \cap \Xi_e^{0,N+2}$. Hence

$$V_{N+1}(\xi^o) \quad = \quad c + \int_S V_N(T(\xi^o,x))\ \bar{f}_\xi(x)\ d\mu(x)$$

and

$$V_{N+2}(\xi^o) \quad = \quad c + \int_S V_{N+1}(T(\xi^o,x))\ \bar{f}_\xi(x)\ d\mu(x) \quad .$$

It follows, by the above two relations, that

$$\begin{aligned} V_{N+1}(\xi^o) - V_{N+2}(\xi^o) \quad &= \quad \int_S (V_N(T(\xi^o,x)) - V_{N+1}(T(\xi^o,x)) \\ &\qquad \bar{f}_\xi(x)\ d\mu(x) \\ &\triangleq \quad \int_S g(x)\ \bar{f}_\xi(x)\ d\mu(x) \quad . \end{aligned}$$

Now $g(x) \geq 0$ by monotonicity of $V_N(\xi)$ w.r.t. N (cf. (3.10)), and $g(x)$ is continuous since $T(\xi^o,x)$ is continuous by condition 4, and $V_N(\xi)$ is continuous w.r.t. ξ in $(0,1)$, being concave and bounded there (c.f. Lemma III.1 and (3.5)). In addition, in view of condition 1, the measure μ with corresponding density $\bar{f}_\xi(.)$ clearly dominates Lebesgue measure ($\mu >> m$). Hence, in view of Lemma III.2, since $g(x^o) > 0$ by

(3.24), we conclude that $V_{N+1}(\xi^o) - V_{N+2}(\xi^o) > 0$, so that, indeed, the problem is not effectively (N+1)-truncated.

To illustrate the results of Theorem III.1, we shall consider the following example:

<u>Example III.1</u> Let x be a normal random variable with densities under $-\theta$ and θ given by

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x+\theta)^2\right) \qquad -\infty < x < +\infty \quad ,$$

$$f_2(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\theta)^2\right) \qquad -\infty < x < +\infty$$

where $\theta > 0$ is known. We have

1. Lebesgue measure $m \sim \mu_i$, where μ_i is the probability measure corresponding to f_i, $i = 1,2$.

2. $$T(\xi,x) = \frac{\xi f_1(x)}{\xi f_1(x) + (1-\xi) f_2(x)}$$

$$= \frac{1}{1 + \frac{1-\xi}{\xi}\frac{f_2(x)}{f_1(x)}}$$

$$= \frac{1}{1 + \frac{1-\xi}{\xi}\exp(2x\theta)} \quad .$$

It is clear that, for any fixed ξ, $T(\xi,x)$ above is continuous w.r.t. x. For $\xi^o = \frac{1}{2}$,

$$T\left(\frac{1}{2}, x\right) = \frac{1}{1+\exp(2x\theta)}, \text{ and for any given } \xi' \in (0,1),$$

there exists $x^o = \frac{1}{2\theta} \ln \frac{1-\xi'}{\xi'}$ such that $T\left(\frac{1}{2}, \frac{1}{2\theta} \ln \frac{1-\xi'}{\xi'}\right) = \xi'$.

The Bayes risk $R(\xi,c)$ as defined in (3.18) may now be written as:

$$R(\xi,c) = c + L \int_{x:\ \ln \frac{f_2(x)}{f_1(x)} > \ln \frac{\xi}{1-\xi}} f_1(x)\,dx + (1-\xi)L \int_{x:\ \ln \frac{f_2(x)}{f_1(x)} \le \ln \frac{\xi}{1-\xi}} f_2(x)\,dx \quad .$$

We have $\ln \frac{f_2(x)}{f_1(x)}$ distributed as normal with mean $-2\theta^2$, variance $4\theta^2$ under $\theta_1 = -\theta$ and $\ln \frac{f_2(x)}{f_1(x)}$ distributed as normal with mean $2\theta^2$, variance $4\theta^2$ under $\theta_2 = \theta$. Hence, at $\xi = \frac{1}{2}$,

$$R\left(\frac{1}{2}, c\right) = c + 2L\,\phi(-\theta) \quad ,$$

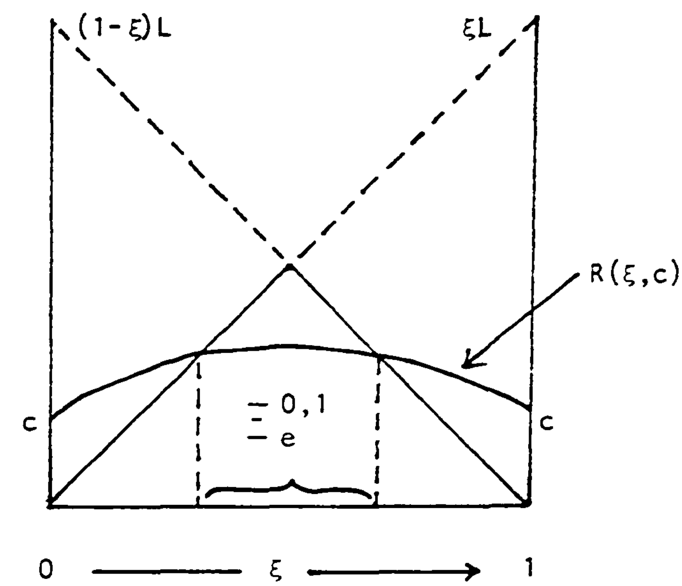
where

$$\phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)dx \quad .$$

Problems such that

$$R\left(\frac{1}{2}, c\right) = c + 2L\,\phi(-\theta) < \frac{L}{2} \tag{3.25}$$

are such that $\Xi_e^{0,1}$ contains a non-null interval, since $R(\xi,c)$ is concave w.r.t. ξ, and also the point $\xi^o = \frac{1}{2}$, (in view of (3.25)) where, as we have already shown, ξ^o does generate (0,1). The general configuration of $R(\xi,c)$ and $\Xi_e^{0,1}$ is as follows:



Hence, by Theorem III.1, this sequential decision problem is effectively non-truncated.

C. A Condition for Effective Truncation

Consider a problem with one source of information, two states of nature, sampling cost c, and arbitrary decision loss $L(a_j,\theta_k)$; $j,k = 1,2$.

Lemma III.3

If $R(\xi,c) \geq V_o(\xi)$, $\forall\xi$, then the sequential decision problem is effectively 0-truncated.

Proof Since $V_1(\xi) = \min\{V_o(\xi), R(\xi,c)\}$ (cf. (3.23)) and $R(\xi,c) \geq V_o(\xi)$, $\forall\xi$ (assumption), it follows that

$$V_1(\xi) = V_o(\xi) \quad , \quad \forall\xi \quad . \tag{3.26}$$

For $N = 2$, (cf. (3.4)),

$$\begin{aligned} V_2(\xi) &= \min \{V_o(\xi),\ c + \int_S V_1(T(\xi,x))\ \bar{f}_\xi(x)\ d\mu(x)\} \\ &= \min \{V_o(\xi),\ c + \int_S V_o(T(\xi,x))\ \bar{f}_\xi(x)\ d\mu(x) \\ &= V_1(\xi) \\ &= V_o(\xi) \quad , \quad \forall \xi \quad . \end{aligned}$$

The second equality follows by $V_1(\xi) = V_o(\xi)$ implies $V_1(T(\xi,x)) = V_o(T(\xi,x))$, $\forall \xi$, $\forall x$, the third equality follows by the fact that the R.H.S. of the second equality is in fact equal to $V_1(\xi)$, and the last equality follows by (3.26).

By finite induction, it is true that, if $V_N(\xi) = V_o(\xi)$, $\forall \xi$, then $V_{N+1}(\xi) = V_o(\xi)$, $\forall \xi$; hence, the sequential decision problem is effectively 0-truncated.

Lemma III.4

A problem with 0 - L decision loss such that $R(\frac{1}{2}, c) \geq \frac{L}{2}$ is effectively 0-truncated.

Proof Since $R(0,c) = R(1,c) = c$ and $R(\frac{1}{2}, c) \geq \frac{L}{2}$, and since $R(\xi,c)$ is a concave function w.r.t. ξ (3), it follows that $R(\xi,c)$ does not lie below $\min \{l_1(\xi), l_2(\xi)\}$, where $l_1(\xi)$ is a line connecting $(0,c)$ and $(\frac{1}{2}, \frac{L}{2})$, and $l_2(\xi)$ is a line connecting $(\frac{1}{2}, \frac{L}{2})$ and $(1,c)$. By relation (3.19), $V_o(\xi) = \min \{\xi L, (1-\xi)L\}$. Hence, $R(\xi,c) \geq V_o(\xi)$, $\forall \xi$; therefore, by Lemma III.3, the sequential decision problem is effectively 0-truncated.

In view of Lemma III.4, if $R(\frac{1}{2}, c) < \frac{L}{2}$, then the problem is not effectively 0-truncated and it is equally true that there exists ξ^* such that $R(\xi^*,c)$ is strictly less than $V_o(\xi^*)$. Then it follows, by concavity of $R(\xi,c)$, that $\Xi_e^{0,1}$, as defined in Section II.C., contains a non-null interval.

Example III.2

Let X be a Bernoulli random variable which takes the values 0 or 1 with densities under θ_1, θ_2 given by

	X	
	0	1
θ_1:	p	q
θ_2:	q	p

The sampling cost is c, and the terminal decision loss is 0 - L. Without loss of generality, we shall let $p < 0.5$. We can verify that the risk $R(\xi,c)$ as defined in (3.18) is as follows:

$$\begin{aligned} R(\xi,c) &= c + \xi L && \text{for } 0 \le \xi \le p \\ &= c + pL && \text{for } p < \xi < 1 - p \\ &= c + (1-\xi)L && \text{for } 1-p \le \xi \le 1 \quad . \end{aligned}$$

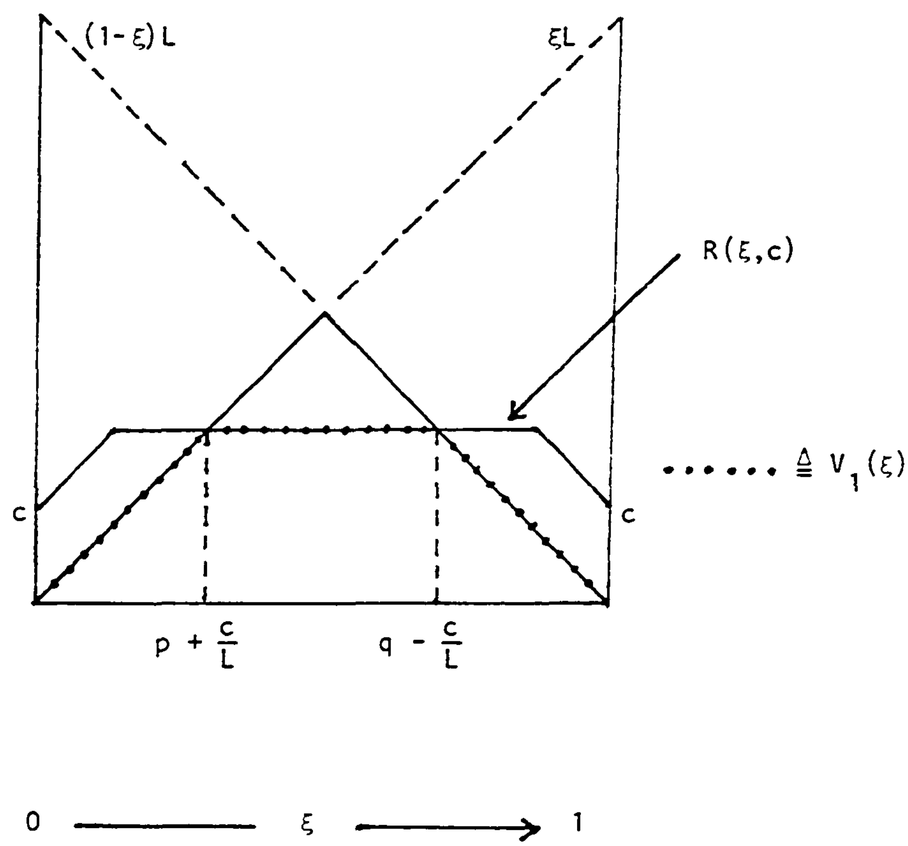
The Bayes risk of the optimal no-data Bayes policy is

$$\begin{aligned} V_o(\xi) &= \xi L && \text{for } 0 \le \xi \le 0.5 \quad , \\ &= (1-\xi)L && \text{for } 0.5 < \xi \le 1 \quad . \end{aligned}$$

Hence, $V_1(\xi)$, as defined in (3.23), is given by

$$
\begin{aligned}
V_1(\xi) &= \xi L && \text{for } 0 \le \xi \le p + \tfrac{c}{L} \quad , \\
&= c + pL && \text{for } p + \tfrac{c}{L} < \xi \le 1 - (p + \tfrac{c}{L}) \quad , \\
&= (1-\xi)L && \text{for } 1 - (p + \tfrac{c}{L}) < \xi \le 1 \quad .
\end{aligned}
$$

A general configuration for $V_o(\xi)$, $R(\xi,c)$ and $V_1(\xi)$ is as follows:



The above problem such that $R(0.5,c) = c + pL < \frac{L}{2}$ is not effectively 0-truncated; otherwise, by Lemma III.4, the problem is effectively 0-truncated.

Suppose that the problem is such that $c + pL < \frac{L}{2}$; then we have $\Xi_e^{0,1} = (p + \frac{c}{L}, q - \frac{c}{L})$. An important characteristic of such $\Xi_e^{0,1}$ is that if there is no $\xi^o \in \Xi_e^{0,1}$ that can generate a point $\xi^* \in \Xi_e^{0,1}$ then the problem is effectively 1-truncated.

Consider $V_2(\xi)$ at $\xi = p + \frac{c}{L}$. We know that the continuation region $\Xi_e^{0,2}$ of policy δ^{*2} contains the continuation region $\Xi_e^{0,1}$ of δ^{*1}. Then δ^{*2} calls for sampling at $\xi = p + \frac{c}{L}$. Hence

$$\begin{aligned} V_2(p + \tfrac{c}{L}) &= c + V_1(T(p + \tfrac{c}{L}, 0))p(X = 0) \\ &\quad + V_1(T(p + \tfrac{c}{L}, 1))p(X = 1) \quad . \end{aligned}$$

For $X = 0$, the posterior $T(p + \frac{c}{L}, 0)$ will move to the left of $\xi = p + \frac{c}{L}$ and hence, will be in the stopping region of δ^{*1}. Hence,

$$V_1(T(p + \tfrac{c}{L}, 0)) = V_o(T(p + \tfrac{c}{L}, 0)) \quad .$$

For $X = 1$, the posterior $T(p + \frac{c}{L}, 1)$ may move into either the continuation or stopping region. If we should impose some conditions so that $T(p + \frac{c}{L}, 1)$ will be in the stopping region, then $V_2(p + \frac{c}{L}, 1)$ may be written in the same format as $V_1(p + \frac{c}{L})$.

Now $T(p + \frac{c}{L}, 1)$ is in the stopping region whenever $T(p + \frac{c}{L}, 1) \geq q - \frac{c}{L}$, or, since

$$T(p + \frac{c}{L}, 1) = \frac{(p + \frac{c}{L})q}{(p + \frac{c}{L})q + (q - \frac{c}{L})p} \quad ,$$

whenever

$$\frac{(p + \frac{c}{L})q}{(p + \frac{c}{L})q + (q - \frac{c}{L})p} \geq q - \frac{c}{L} \quad \text{iff}$$

$$(p + \frac{c}{L})q \geq (q - \frac{c}{L})(p + \frac{c}{L})q + (q - \frac{c}{L})^2 p \quad \text{iff}$$

$$(p + \frac{c}{L})^2 q - (q - \frac{c}{L})^2 p \geq 0 \quad \text{iff}$$

$$\frac{(p + \frac{c}{L})^2}{(q - \frac{c}{L})^2} \geq \frac{p}{q} \quad .$$

Solving for c, we have

$$c \geq L\left[\frac{\sqrt{p}}{\sqrt{p} + \sqrt{q}} - p\right] \quad .$$

If $c \geq L\left[\frac{\sqrt{p}}{\sqrt{p} + \sqrt{q}} - p\right]$, then

$$V_2(p + \frac{c}{L}) = c + V_o(T(p + \frac{c}{L}, 0))p(X = 0) + V_o(T(p + \frac{c}{L}, 1))p(X = 1)$$

$$= V_1(p + \frac{c}{L}) \quad .$$

In fact, the condition $c \geq L\left[\frac{\sqrt{p}}{\sqrt{p} + \sqrt{q}} - p\right]$ implies that for any

$p + \frac{c}{L} \le \xi \le q - \frac{c}{L}$ we have both $T(\xi,0)$ and $T(\xi,1)$ are in the stopping region of δ^{*1}. Hence,

$$V_2(\xi) = V_1(\xi) \quad , \quad p + \frac{c}{L} \le \xi \le q - \frac{c}{L} \quad .$$

Therefore,

$$V_2(\xi) = V_1(\xi) \quad , \quad \forall \xi \quad .$$

By induction, we can verify that

$$V_N(\xi) = V_1(\xi) \quad , \quad \forall N \quad , \quad \forall \xi \quad .$$

Hence, the problem is effectively 1-truncated. Thus, we may state that the following lemma with proof has already been given in the above discussion.

<u>Lemma III.5</u>

For the problem set-up in Example III.2, if

1. $c + pL < \frac{L}{2}$

and

2. $c \ge L\left[\dfrac{\sqrt{p}}{\sqrt{p} + \sqrt{q}} - p\right]$,

Then the problem is effectively 1-truncated.

IV. SOURCE SELECTION AND EXPENDABILITY IN THE CASE OF TWO SOURCES OF INFORMATION

A. Introduction

It may frequently happen (3) that a statistician finds that there are two experiments or sources of information available to him which he might perform to guide him in reaching decisions. Thus, he is faced with a preliminary decision regarding which experiment to perform. If he admits the possibility of performing more than one, then the question arises of how many and which experiments to perform, and in what order. On the other hand, he may ask himself if he really needs two different kinds of experiments. If the answer is negative, then the problem is simplified, falling into the domain of the subject matter of the previous chapter, since he is then dealing with just one source of information. The question of whether two different kinds of experiments are needed comes under the topic of source expendability; the question of when and which one of the two experiments to perform comes under the topic of source selection, and we shall focus on the former.

Let X, Y be two real-valued random variables, associated with two sources of information, whose two sample spaces S coincide, and suppose that the sampling cost of X (resp. Y) is c_x (resp. c_y). Suppose that X and Y have densities, with respect to a common σ-finite measure μ, under two states of nature (θ_1, θ_2) as shown by

$$\begin{array}{ccc} & X & Y \\ \theta_1: & f_1(x) & g_1(y) \end{array}$$

$$\theta_2: \qquad f_2(x) \qquad g_2(y)$$

where

$$f_1(s),\ f_2(s),\ g_1(s),\ g_2(s) \ > \ 0 \quad , \quad s \in S \quad . \tag{4.1}$$

Define

$$T_x(\xi,x) \ = \ \frac{\xi f_1(x)}{\xi_1 f_1(x) + (1-\xi) f_2(x)} \quad , \tag{4.2}$$

$$T_y(\xi,y) \ = \ \frac{\xi g_1(y)}{\xi g_1(y) + (1-\xi) g_2(y)} \quad , \tag{4.3}$$

$$\bar{f}_\xi(x) \ = \ \xi f_1(x) + (1-\xi) f_2(x) \quad , \tag{4.4}$$

$$\bar{g}_\xi(y) \ = \ \xi g_1(y) + (1-\xi) g_2(y) \quad . \tag{4.5}$$

Assuming 0 - L terminal decision loss, it is one of our interests to find conditions to be imposed on the f_i's, g_i's, c_x, c_y and L in order to draw the conclusion that a particular source out of the two is "expendable", the precise definition of this term being given in Definition IV.2 below.

Let $V^2(\xi)$ be the Bayes risk of the optimal unrestricted Bayes policy, and let $V_o^2(\xi)$ be the Bayes risk of the optimal no-data Bayes policy, where the superscript "2" indicates that we are dealing with the case of two sources of information. We also let $V_x^2(\xi)$ (resp. $V_y^2(\xi)$) be the Bayes risk for sampling X (resp. Y) first and then pursuing the optimal Bayes policy corresponding to $V^2(\xi)$ thereafter. As shown in (10), in analogy to the case of one source of information, it is readily verified that $V^2(\xi)$ satisfies the following functional equation:

$$V^2(\xi) = \min \{V_o^2(\xi),\ c_x + \int_S V^2(T_x(\xi,s))\ \bar{f}_\xi(s)\ d\mu(s),$$

$$c_y + \int_S V^2(T_y(\xi,s))\ \bar{g}_\xi(s)\ d\mu(s)\} \quad , \qquad (4.6)$$

We note that $V_o^2(\xi)$ is in fact equal to $V_o(\xi)$, as defined in the case of one source of information, whenever $V_o^2(\xi)$ and $V_o(\xi)$ are computed for the same final decision loss L. The reason for keeping the notation $V_o^2(\xi)$ is to make the relation (4.6) refer consistently to the availability of two sources. It is clear that

$$V_x^2(\xi) = c_x + \int_S V^2(T_x(\xi,s))\ \bar{f}_\xi(s)\ d\mu(s) \quad , \qquad (4.7)$$

and

$$V_y^2(\xi) = c_y + \int_S V^2(T_y(\xi,s))\ \bar{g}_\xi(s)\ d\mu(s) \quad . \qquad (4.8)$$

In view of (4.7) and (4.8), the relation (4.6) may be written as

$$V^2(\xi) = \min \{V_o^2(\xi),\ V_x^2(\xi),\ V_y^2(\xi)\} \quad . \qquad (4.9)$$

<u>Definition IV.1</u>

The continuation region Ξ_c is defined to be

$$\Xi_c = \{\xi \in \Xi :\ V^2(\xi) = \min \{V_x^2(\xi),\ V_y^2(\xi)\}\} \quad . \quad (4.10)$$

Definition IV.2

In a Bayesian sequential decision problem of type F with two sources of information with constant sampling costs, two states of nature and arbitrary final decision loss, source Y is said to be expendable if

$$V_x^2(\xi) \le V_y^2(\xi) \quad , \quad \xi \in \Xi_c \ , \tag{4.11}$$

where Ξ_c is as defined in (4.10).

Note IV.1

In view of (4.6) and (4.9), if source Y is expendable then

$$V^2(\xi) = \min \{V_o^2(\xi),\ c_x + \int_S V^2(T_x(\xi,s))\ \bar{f}_\xi(s)\ d\mu(s)\} \quad ,$$

which is analogous to (3.15), since $T_x(\xi,x)$ is simply $T(\xi,x)$ for the one source case, and $V_o^2(\xi)$ is simply $V_o(\xi)$.

Note IV.2

Suppose we are only given a recipe of the optimal Bayes risk $V^2(\xi)$, $\forall\xi$, without knowing the optimal Bayes policy, and we are able to compute $V_x^2(\xi)$, $V_y^2(\xi)$, $\forall\xi$. If it is the case that source Y is expendable by Definition IV.2, then, in view of Note IV.1, we know that the given $V^2(\xi)$ is identical to the optimal Bayes risk of the policy using only X.

The topic of this chapter, as indicated by the heading, concerns the idea of source expendability; we discuss this idea, in particular, in the context of the normal and binomial case in Section IV.C. Section IV.D. is devoted to showing that a natural expendability condition in the non-sequential case (7) is extendable to a special sequential case. Section IV.E. will cover a discussion of the truncated sequential case.

B. Likelihood Ratios

For the modeling of the first of two information sources under two states of nature, consider two probability spaces $(S,\mathcal{B},\mu_1)$ and $(S,\mathcal{B},\mu_2)$, where S is a measurable subspace of the real line, $\mathcal{B}$ a σ-field of subsets of S, and μ_1, μ_2 are probability measures. It will be useful below to be able to deal with densities corresponding to μ_1, μ_2. Hence, consider the dominating measure

$$\mu_{12} = \mu_1 + \mu_2$$

and, by means of the Radon-Nikodym theorem, define two densities f_1 and f_2, respectively of μ_1 and μ_2, with respect to μ_{12}.

We shall require as well that μ_1 and μ_2 be equivalent, i.e., that

$$\mu_2\{x: \ f_2(x) = 0, \ f_1(x) > 0\} = 0 \quad ,$$

$$\mu_1\{x: \ f_1(x) = 0, \ f_2(x) > 0\} = 0 \quad ,$$

from which it follows that $\frac{f_1(x)}{f_2(x)}$ is finite-valued with probability 1 under μ_2, and $\frac{f_2(x)}{f_1(x)}$ is finite-valued with probability 1 under μ_1. Under these conditions it is possible to speak of the distribution function $F_{12}(t)$ of $\frac{f_1(x)}{f_2(x)}$ under μ_2 and the distribution function $F_{21}(t)$ of $\frac{f_2(x)}{f_1(x)}$ under μ_1. Note that, if μ_1 and μ_2 are absolutely continuous with respect to a third measure μ, such as Lebesgue measure or counting measure, the Radon-Nikodym derivatives g_1 and g_2 of μ_1 and μ_2 with respect to μ may be substituted for f_1 and f_2 in the above ratios.

It is useful for the analysis in the next section and the rest of the chapter, in order to model a second source of information under two states of nature, to introduce another random variable Y and its corresponding probability spaces $(S,\mathcal{B},\tau_1)$ and $(S,\mathcal{B},\tau_2)$. A similar development in terms of introducing $\tau_{12} = \tau_1 + \tau_2$, defining densities g_1 and g_2 respectively of τ_1 and τ_2 with respect to τ_{12}, distribution function $G_{12}(u)$ of $\frac{g_1(y)}{g_2(y)}$ under τ_2, and distribution function $G_{21}(u)$ of $\frac{g_2(y)}{g_1(y)}$ under τ_1 may be carried out. It may happen, as in Example IV.1 and Example IV.2 below, that in certain Bayesian sequential decision problems $F_{12}(u) = G_{12}(u)$ and $F_{21}(u) = G_{21}(u), \forall u$.

Example IV.1

Suppose that X and Y have binomial distributions under both states of nature with parameters given by

	X	Y
θ_1:	p	q
θ_2:	q	p

$$p + q = 1 \quad .$$

It is clear in this case that

$$F_{12}(u) = G_{12}(u) \quad ,$$

and

$$F_{21}(u) = G_{21}(u) \quad , \quad \forall u \quad .$$

Example IV.2

Suppose that X and Y have normal distributions under both states of nature with parameters given by

$$\begin{array}{lll} & X & Y \\ \theta_1: & N(-\mu,\sigma^2) & N(-\eta,\nu^2) \\ \theta_2: & N(\mu,\sigma^2) & N(\eta,\nu^2) \end{array} ,$$

where the first component in the parentheses stands for mean and the second one stands for variance. It is true that $F_{12}(u) = G_{12}(u)$ and $F_{21}(u) = G_{21}(u)$, $\forall u$, provided that $\frac{\sigma}{\mu} = \frac{\nu}{\eta}$.

Definition IV.3

(Condition A) The densities f_1, f_2 of X and g_1, g_2 of Y are said to satisfy the condition A if

$$F_{12}(u) = G_{12}(u) \quad \text{and} \quad F_{21}(u) = G_{21}(u) \quad , \quad \forall u \quad .$$

A related condition, arising in connection with the sort of symmetry analyzed in Chapter V, is given in

Definition IV.4

(Condition B) The densities f_1, f_2 are said to satisfy condition B if $F_{12}(u) = F_{21}(u)$, $\forall u$.

We note that Definition IV.3 applies to four densities while Definition IV.4 applies to only two.

C. Source Expendability

In this section, we shall specialize the Bayesian sequential decision problem to the case of two states of nature, two sources of information X, Y, sampling costs c_x, c_y, and final decision loss 0 - L. Suppose that X and Y have the same sample space and densities with respect to a common σ-finite measure μ given by

	X	Y
θ_1:	$f_1(x)$	$g_1(y)$
θ_2:	$f_2(x)$	$g_2(y)$.

We assume as well condition (4.1).

Conditions for particular sources to be expendable are now explored in the context of relation (4.10). Consider:

$$V_x^2(\xi) = c_x + \int_S V^2(T_x(\xi,s))\, \bar{f}_\xi(s)\, d\mu(s) \quad ,$$

which can be rewritten as

$$V_x^2(\xi) = c_x + \xi \int_S V^2\left[\frac{1}{1 + \frac{1-\xi}{\xi}\frac{f_2(s)}{f_1(s)}}\right] f_1(s)\, d\mu(s)$$

$$+ (1-\xi) \int_S V^2\left[\frac{1}{1 + \frac{1-\xi}{\xi}\frac{f_2(s)}{f_1(s)}}\right] f_2(s)\, d\mu(s)$$

$$\triangleq c_x + \xi E_{x,\theta_1}\left[V^2(\phi_\xi\left(\frac{f_2(x)}{f_1(x)}\right))\right] + (1-\xi) E_{x,\theta_2}\left[V^2(\psi_\xi\left(\frac{f_1(x)}{f_2(x)}\right))\right], \tag{4.11}$$

where

$$\phi_\xi(t) = \frac{1}{1 + \frac{1-\xi}{\xi} t} ,$$

and

$$\psi_\xi(t) = \frac{1}{1 + \frac{1-\xi}{\xi} \frac{1}{t}} .$$

Similarly,

$$V_y^2(\xi) = c_y + \int_S V^2(T_y(\xi,s)) \bar{g}_\xi(s) \, d\mu(s)$$

$$\triangleq c_y + E_{y,\theta_1} V^2(\phi_\xi(\frac{g_2(s)}{g_1(s)}))$$

$$+ (1-\xi)E_{y,\theta_2} V^2(\psi_\xi(\frac{g_1(s)}{g_2(s)})) . \tag{4.12}$$

The R.H.S. of relations (4.11) and (4.12) indicate that we may be able to impose condition on the distribution function of the likelihood ratios and sampling costs so that a particular source is expendable.

Lemma IV.1

If X and Y are such that condition A is satisfied, then $c_x \le c_y$ implies that Y is expendable.

Proof In view of relations (4.11), (4.12) and condition A, we have

$$E_{x,\theta_1}\left[V^2(\phi_\xi (\frac{f_2(s)}{f_1(s)})) \right] + (1-\xi)E_{x,\theta_2}\left[V^2(\psi_\xi (\frac{f_1(s)}{f_2(s)})) \right] =$$

$$E_{y,\theta_1}\left[V^2(\phi_\xi (\frac{g_2(s)}{g_1(s)})) \right] + (1-\xi)E_{y,\theta_2}\left[V^2(\phi_\xi (\frac{g_1(s)}{g_2(s)})) \right] \quad . \tag{4.13}$$

In view of (4.11), (4.12), (4.13) and $c_x \le c_y$, we have

$$V_x^2(\xi) \le V_y^2(\xi) \quad , \quad \forall \xi \quad ,$$

so that Y is expendable.

Note that both Example IV.1 and IV.1 satisfy that part (condition A) of the given of Lemma IV.1 not referring to sampling costs.

It has been proved (3,23,30) in the case of one source of information with a finite number of states of nature that the optimal Bayes risk is concave w.r.t. $\underline{\xi}$. It is also true (cf. (10)) in the case of two sources of information that the optimal Bayes risk $V^2(\xi)$, as defined in (4.6), is concave w.r.t. $\underline{\xi}$. This fact is used in the remainder of this section, which treats the case where X and Y both are distributed as binomial random variables satisfying condition B with densities given by

	X		Y	
	0	1	0	1
θ_1:	p	q	u	v
θ_2:	q	p	v	u .

We note that X and Y satisfying condition B insure that all V functions will be symmetric in ξ about the point $\frac{1}{2}$. This symmetry is appealed to repeatedly in the remainder of this section.

Consider $V_x^2(\xi)$, as defined in (4.7), which may be written now as

$$V_x^2(\xi) = c_x + V^2\left(\frac{\xi p}{\xi p + (1-\xi)q}\right)(\xi p + (1-\xi)q) + V^2\left(\frac{\xi q}{\xi q + (1-\xi)p}\right)(\xi q + (1-\xi)p) \quad , \tag{4.14}$$

and $V_y^2(\xi)$, as defined in (4.8), which may be written now as

$$V_y^2(\xi) = c_y + V^2\left(\frac{\xi u}{\xi u + (1-\xi)v}\right)(\xi u + (1-\xi)v) + V^2\left(\frac{\xi v}{\xi v + (1-\xi)u}\right)(\xi v + (1-\xi)u) \quad . \tag{4.15}$$

We shall let $1_{x,\xi}(t)$ denote the straight line joining

$\left(\frac{\xi p}{\xi p + (1-\xi)q}, V^2\left(\frac{\xi p}{\xi p + (1-\xi)q}\right)\right)$ and $\left(\frac{\xi q}{\xi q + (1-\xi)p}, V^2\left(\frac{\xi q}{\xi q + (1-\xi)p}\right)\right)$

and analogously for $1_{y,\xi}(t)$, replacing p by u and q by v. It follows, letting $t = \xi$, that

$$1_{x,\xi}(\xi) = V^2\left(\frac{\xi p}{\xi p + (1-\xi)q}\right)(\xi p + (1-\xi)q) + V^2\left(\frac{\xi q}{\xi q + (1-\xi)p}\right)(\xi q + (1-\xi)p) \quad , \tag{4.16}$$

and

$$1_{y,\xi}(\xi) = V^2\left(\frac{\xi u}{\xi u + (1-\xi)v}\right)(\xi u + (1-\xi)v) + V^2\left(\frac{\xi v}{\xi v + (1-\xi)u}\right)(\xi v + (1-\xi)u) \quad . \tag{4.17}$$

<u>Lemma IV.2</u>

Let X and Y be two random variables. Suppose that $a \le c < d \le b$ and

$$P(X = a) = 1 - P(X = b) = \alpha$$

$$P(Y = c) = 1 - P(Y = d) = \beta \quad , \quad 0 < \alpha, \beta < 1 \quad .$$

For any concave function $h(t)$, if

$$E(X) = E(Y) = k \quad , \tag{4.18}$$

then

$$E(h(X)) \leq E(h(Y)) \quad .$$

<u>Proof</u> Let $1_x(t)$ be the straight line joining $(a, h(a))$ and $(b, h(b))$, and also let $1_y(t)$ be the straight line joining $(c, h(c))$ and $(d, h(d))$. Since h is concave and $a \leq c < d \leq b$, it follows that $h(c) \geq 1_x(c)$ and $h(d) \geq 1_x(d)$, so that

$$1_x(t) \leq 1_y(t) \quad , \quad t \in [c,d] \quad . \tag{4.19}$$

The condition (4.17) now implies that

$$c \leq E(X) \leq d \quad . \tag{4.20}$$

Since $1_x(t)$, $1_y(t)$ are linear, we have now that

$$\begin{aligned} E(h(X)) &= \alpha h(a) + (1-\alpha)h(b) \\ &= 1_x(E(X)) \\ &\leq 1_y(E(X)) \\ &= 1_y(E(Y)) \\ &= \beta h(c) + (1-\beta)h(d) \\ &= E(h(Y)) \quad . \end{aligned}$$

The first equality follows by definition, the second equality follows by $h(a) = 1_x(a)$, $h(b) = 1_x(b)$ and $1_x(t)$ linear, the first inequality follows by (4.19) and (4.18), the third equality follows by (4.17), the fourth equality follows by $h(c) = 1_y(c)$, $h(d) = 1_y(d)$ and $1_y(t)$ linear, and, finally, the fifth equality follows by definition.

By consulting (4.6), (4.14), (4.15), (4.16), (4.17), and Lemma IV.2, the proof of the next lemma is clear and is thus omitted.

Lemma IV.3

If $c_x = c_y$ and $0 \le p < u \le \frac{1}{2}$ then y is expendable.

The following lemma is drawn from Borwanker, David and Ingwell (4).

Lemma IV.4

If

1. $p < \frac{c_x}{L}$ and either
2. $c_y + uL > \frac{L}{2}$ or
3. $\frac{uv}{(1-2u)L}[(1-2p)L - 2c_x]^2 < c_y - Lv - (c_x - Lq)$,

then Y is expendable.

The next lemma gives conditions analogous but alternative to those of Lemma IV for Y to be expendable.

Lemma IV.5

If

1. $c_x \le c_y$
2. $u, p < \frac{1}{2}$

and

3. $c_y + \min\{c_x[\frac{c_y}{L}u + (1 - \frac{c_y}{L})v], c_y u\}$

$$+ \min \{c_x[\frac{c_y}{L} v + (1 - \frac{c_y}{L})u],\ c_y v,\ (L - c_y)u\}$$

$$\geq\ c_x + LP \quad ,$$

then Y is expendable.

<u>Proof</u> Let $V^2(\xi)$, $V_x^2(\xi)$ and $V_y^2(\xi)$ be as defined in (4.6), (4.7) and (4.8) respectively. Since X and Y satisfy condition B and therefore lead to symmetric V functions, i.e., $V^2(\xi) = V^2(1-\xi)$, $V_x^2(\xi) = V_x^2(1-\xi)$ and $V_y^2(\xi) = V_y^2(1-\xi)$, we may consider their properties just on $[0, \frac{1}{2}]$.

Let ξ^* be the greatest lower bound of ξ's such that the optimal Bayes policy calls for sampling from Y; then, in view of the concavity of $V_y^2(\xi)$, it is true that $\xi^* \geq \frac{c_y}{L}$. We assume, without loss of generality, that $\frac{c_y}{L} < \frac{1}{2}$. At $\xi = \frac{c_y}{L}$, we have

$$V_y^2(\frac{c_y}{L}) \;=\; c_y + V^2\,[T_y(\frac{c_y}{L}, 0)](\frac{c_y}{L} u + (1 - \frac{c_y}{L})v)$$

$$+ V^2\,[T_y(\frac{c_y}{L}, 1)](\frac{c_y}{L} v + (1 - \frac{c_y}{L})u) \quad .$$

Consider two possible cases concerning $T_y(\frac{c_y}{L}, 0)$:

1. If $T_y(\frac{c_y}{L}, 0)$ lies in the continuation region, then, by relation (4.6) and $c_x \leq c_y$, $V^2[T_y(\frac{c_y}{L}, 0)] \geq c_x$.

2. If $T_y(\frac{c_y}{L}, 0)$ lies in the termination region, then

$$V^2[T_y(\frac{c_y}{L}, 0)] \;=\; \frac{\frac{c_y}{L} u}{\frac{c_y}{L} u + (1 - \frac{c_y}{L})v} \cdot L \quad ,$$

since both $\frac{c_y}{L}$ and u are less than $\frac{1}{2}$ so that

$$\frac{\frac{c_y}{L} u}{\frac{c_y}{L} u + (1 - \frac{c_y}{L})v} < \frac{1}{2} \quad .$$

Similarly, consider three possible cases concerning $T_y(\frac{c_y}{L}, 1)$:

1. If $Ty(\frac{c_y}{L}, 1)$ lies in the continuation region then $V^2[T_y(\frac{c_y}{L}, 1)] \geq c_x$.

2. If $T_y(\frac{c_y}{L}, 1)$ lies in the termination region then there are two possibilities. Either $T_y(\frac{c_y}{L}, 1) \leq \frac{1}{2}$ or $T_y(\frac{c_y}{L}, 1) > \frac{1}{2}$.

2.1. If $T_y(\frac{c_y}{L}, 1) \leq \frac{1}{2}$; that is, $\frac{c_y}{L} v \leq (1 - \frac{c_y}{L})u$, then

$$V^2[T_y(\frac{c_y}{L}, 1)] = \frac{\frac{c_y}{L} v}{\frac{c_y}{L} v + (1 - \frac{c_y}{L}) u} \cdot L \quad .$$

2.2. If $T_y(\frac{c_y}{L}, 1) > \frac{1}{2}$; that is, $\frac{c_y}{L} v > (1 - \frac{c_y}{L})u$, then

$$V^2[T_y(\frac{c_y}{L}, 1)] = \frac{(1 - \frac{c_y}{L})u}{\frac{c_y}{L} v + (1 - \frac{c_y}{L})u} \cdot L \quad .$$

Combining all possible cases, we have

$$\begin{aligned} V_y^2(\frac{c_y}{L}) \geq\ & c_y + \min \{c_x(\frac{c_y}{L} u + (1 - \frac{c_y}{L})v), c_y u\} \\ & + \min \{c_x(\frac{c_y}{L} v + (1 - \frac{c_y}{L})u), c_y v, (L - c_y)u\} \quad . \end{aligned} \tag{4.20}$$

It is clear that

$$V_x^2\left(\frac{1}{2}\right) \leq c_x + Lp \quad . \tag{4.21}$$

Hence

$$V_y^2\left(\frac{c_y}{L}\right) \geq V_x^2\left(\frac{1}{2}\right) \quad , \tag{4.22}$$

by (4.20), (4.21) and condition 3 of the lemma.

We know that the set of ξ's, if any, such that the optimal Bayes policy calls for sampling Y first is a subset of the interval $[\frac{c_y}{L}, 1 - \frac{c_y}{L}]$. In view of (4.22) and concavity of V_y^2 we have shown that, for any $\xi \varepsilon [\frac{c_y}{L}, 1 - \frac{c_y}{L}]$,

$$V_x^2(\xi) \leq V_y^2(\xi) \quad .$$

Therefore, Y is expendable.

The next lemma concerns itself with conditions for non-expendability of a source which are partly related to previous conditions for expendability of a source.

<u>Lemma IV.6</u>

If $u < p < \frac{1}{2}$ and $c_x + pL \leq c_y + uL < \frac{L}{2}$, then the optimal Bayes policy is to sample X first at $\xi = \frac{1}{2}$.

<u>Proof</u> It is clear that

$$V_x^2\left(\frac{1}{2}\right) = c_x + V^2(p) \quad ,$$

and

$$V_y^2(\tfrac{1}{2}) = c_y + V^2(u) \quad .$$

Therefore,

$$\begin{aligned} V_x^2(\tfrac{1}{2}) - V_y^2(\tfrac{1}{2}) &= c_x - c_y + V^2(p) - V^2(u) \\ &\le c_x - c_y + (p - u)L \\ &\le 0 \quad . \end{aligned}$$

The first inequality comes from the concavity of $V^2(\xi)$ and $u < p < \frac{1}{2}$, the second inequality comes from the condition $c_x + pL \le c_y + uL$.

We close this section by noting that the condition $u < p < \frac{1}{2}$ of Lemma IV.6 is essential. If $p < u < \frac{1}{2}$, then the condition $c_x + pL \le c_y + uL$ reflects that there is a possibility that $c_x > c_y$ and we may have a case where X is expendable, by reasoning that the higher-cost source is less desirable in the long run, compared with wrong-decision loss, than in the short run. An example is given for the above discussion in Section IV.E.

D. Uniform Risk Inequalities

Consider the situation of Section IV.C., with $L = 1$. Let $R_z(\xi,c_z)$ be the risk of the fixed-sample-size Bayes policy, w.r.t. the prior ξ, using one observation from source Z, Z = X or Y. The Bayes risk (28), w.r.t. ξ, using one costless X ($c_x = 0$) is given by

$$R_x(\xi,0) = \xi \int_A f_1(x)\, d\mu(x) + (1 - \xi) \int_B f_2(x)\, d\mu(x) \qquad , \quad (4.23)$$

where

$$A = \{x: \frac{f_2(x)}{f_1(x)} > \frac{\xi}{1-\xi}\}, \quad B = \{x: \frac{f_2(x)}{f_1(x)} \le \frac{\xi}{1-\xi}\} \quad ,$$

and similarly for Y.

Definition IV.5

Source X is said to be at least as informative as source Y if $R_x(\xi,0) \le R_y(\xi,0)$, $\forall\xi$, which we denote below by $R_x \le R_y$.

Bradt and Karlin (7) give general conditions for $R_x \le R_y$, some of which are related to those obtained by Blackwell (2). They point out as well that source X is to be used consistently whenever $R_x \le R_y$, in order to have the least Bayes risk in the case when a fixed number of total observations is specified and the X and Y are costless, a state of affairs closely related to the notion of expendability introduced in Section IV.C. We now indicate why $R_x \le R_y$ does fulfill this role. Let $R_{z_1,z_2}(\xi)$ be Bayes risk using two observations Z, Z = X or Y, with z_1 the first observation and z_2 the second observation, when sampling cost is zero ($c_x = c_y = 0$).

Lemma IV.7

If $R_x \le R_y$ then

$$R_{x,x}(\xi) \le R_{x,y} = R_{y,x}(\xi) \le R_{y,y}(\xi) \quad , \quad \forall\xi \quad .$$

Proof It is clear that $R_{x,y}(\xi) = R_{y,x}(\xi)$, $\forall\xi$. Consider now $R_{x,x}(\xi)$ and $R_{x,y}(\xi)$:

$$R_{x,x}(\xi) = \int_S [T_x(\xi,s)\int_{t:\ \frac{f_2(t)}{f_1(t)} > \frac{T_x(\xi,s)}{1-T_x(\xi,s)}} f_1(t)d\mu(t) + (1-T_x(\xi,s))\int_{t:\ \frac{f_2(t)}{f_1(t)} \le \frac{T_x(\xi,s)}{1-T_x(\xi,s)}} f_2(t)d\mu(t)]\bar{f}_\xi(s)d\mu(s) \quad , \tag{4.24}$$

$$R_{x,y}(\xi) = \int_S [T_x(\xi,s)\int_{t:\ \frac{g_2(t)}{g_1(t)} > \frac{T_x(\xi,s)}{1-T_x(\xi,s)}} g_1(t)d\mu(t) + (1-T_x(\xi,s))\int_{t:\ \frac{g_2(t)}{g_1(t)} \le \frac{T_x(\xi,s)}{1-T_x(\xi,s)}} g_2(t)d\mu(t)]\bar{f}_\xi(s)d\mu(s) \quad , \tag{4.25}$$

where T is as defined in Section IV.A. The term in the parentheses [.] of (4.24) is uniformly less than that of (4.25) w.r.t. s; hence $R_{x,x}(\xi) \le R_{x,y}(\xi)$, $\forall\xi$. Similarly, we have $R_{y,x}(\xi) \le R_{y,y}(\xi)$, $\forall\xi$. Therefore,

$$R_{x,x}(\xi) \le R_{x,y}(\xi) = R_{y,x}(\xi) \le R_{y,y}(\xi) \quad , \quad \forall\xi \quad .$$

Arguments analogous to those of Lemma IV.7 enable us to say that if $R_x \le R_y$ then, for any specified fixed number of total observations, consistent use of X gives the least Bayes risk.

Since it is almost impossible in a real-world problem to experiment at no cost, it seems useful to introduce sampling costs into the analysis. It is trivial to verify that the results in Bradt and Karlin (7) remain true whenever the sampling costs of both sources are equal. It is our main interest in this section to point out that the uniform inequality of Bayes risk $R_x \le R_y$ for any binomial problem does not only imply that source Y need not be used in the fixed-sample-size case but also implies the expendability of source Y in the sequential case when sampling costs are equal and the final decision loss is 0 - L.

Consider the following binomial situation:

	X		Y	
	0	1	0	1
θ_1:	p	q	p'	q'
θ_2:	u	v	u'	v'

$c_x = c_y = c$, and loss is 0 - L.

Without loss of generality, suppose that L = 1, $u \neq p$, $u' \neq p'$, $\frac{u}{p} < 1 < \frac{v}{q}$, and $\frac{u'}{p'} < 1 < \frac{v'}{q'}$. As defined in (4.23), we have

$$R_x(\xi,0) = \xi \sum_A f_1(x) + (1 - \xi) \sum_B f_2(x) \quad ,$$

where

$$A = \left\{ x: \frac{f_2(x)}{f_1(x)} > \frac{\xi}{1-\xi} \right\} \quad , \quad B = \left\{ x: \frac{f_2(x)}{f_1(x)} \leq \frac{\xi}{1-\xi} \right\} \quad ,$$

and similarly for Y. By varying the values of $\frac{\xi}{1-\xi}$, it can be verified that

$$\left. \begin{aligned} R_x(\xi,0) &= \xi && \text{for } 0 \leq \frac{\xi}{1-\xi} < \frac{u}{p} \\ &= \xi q + (1-\xi)u && \text{for } \frac{u}{p} \leq \frac{\xi}{1-\xi} < \frac{v}{q} \\ &= 1 - \xi && \text{for } \frac{\xi}{1-\xi} \geq \frac{v}{q} \quad . \end{aligned} \right\} \tag{4.26}$$

Similarly,

$$
\left.\begin{aligned}
R_y(\xi,0) &= \xi && \text{for } 0 \le \frac{\xi}{1-\xi} < \frac{u'}{p'} \\
&= \xi q' + (1-\xi)u' && \text{for } \frac{u'}{p'} \le \frac{\xi}{1-\xi} < \frac{v'}{q'} \\
&= 1 - \xi && \text{for } \frac{\xi}{1-\xi} \ge \frac{v'}{q'} \quad .
\end{aligned}\right\} \qquad (4.27)
$$

We note that both $R_x(\xi,0)$ and $R_y(\xi,0)$ are continuous w.r.t. ξ and consist of three piecewise linear functions (cf. (4.26) and (4.27)). The following is useful for the later development.

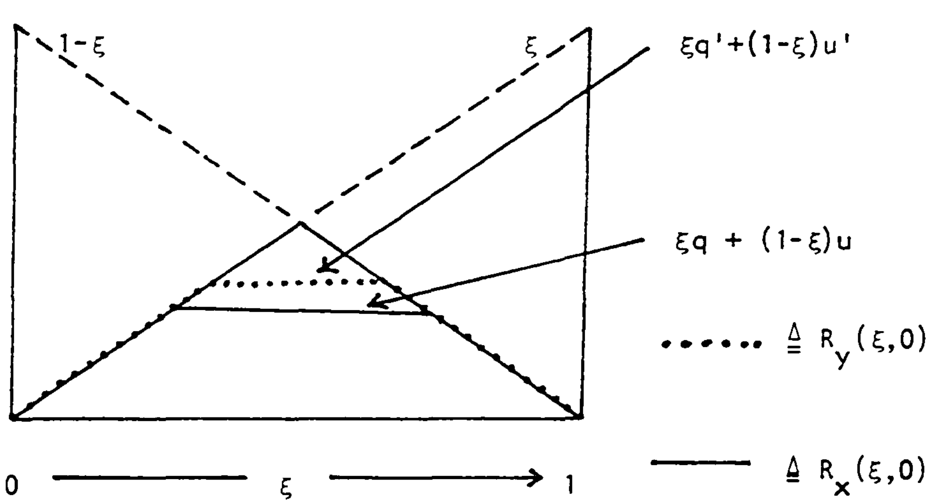
Lemma IV.8

A sufficient condition for

$$
\frac{u}{p} \le \frac{u'}{p'} < 1 < \frac{v'}{q'} \le \frac{v}{q} \qquad (4.28)
$$

is $R_x \le R_y$.

Proof The general behavior of $R_x(\xi,0)$ and $R_y(\xi,0)$ in the binomial case, when $R_x \le R_y$, is as follows



It is clear that, unless $R_x(\xi,0) = R_y(\xi,0)$, $\forall\xi$, in which case inequalities obtain in (4.28), there is a $\xi^* < \frac{1}{2}$ or a $\xi^* > \frac{1}{2}$ such that

$$R_x(\xi^*) < R_y(\xi^*) \quad . \tag{4.29}$$

Without loss of generality assume the former, in which case, in view of (4.27), (4.29) holds as well for ξ^* such that

$$\frac{\xi^*}{1 - \xi^*} = \frac{u'}{p'} \quad . \tag{4.30}$$

But (4.29), (4.26) and (4.27) imply that

$$\xi^* q + (1 - \xi^*)u \leq \xi^* \quad ,$$

or

$$\frac{u}{p} \leq \frac{\xi^*}{1 - \xi^*} \text{ , which implies } \frac{u}{p} \leq \frac{u'}{p'} \text{ by (4.30).}$$

Similarly, we can show that $\frac{v'}{q'} \leq \frac{v}{q}$.

Theorem IV.1

If $R_x \leq R_y$, $c_x = c_y = c$, loss is 0 - L, and both X and Y are binomial, then Y is expendable in the unrestricted sequential sense.

Proof Suppose, without loss of generality, that the densities of X and Y are described in terms of p, q, u, v and p', q', u', v', respectively, and also $\frac{u}{p} < 1 < \frac{v}{q}$, $\frac{u'}{p'} < 1 < \frac{v'}{q'}$.

The condition $R_x \leq R_y$ and Lemma IV.8 imply that

$$\frac{u}{p} \leq \frac{u'}{p'} < 1 < \frac{v'}{q'} \leq \frac{v}{q} \quad . \tag{4.31}$$

Since the posterior probability distribution in favor of θ_1 is monotone non-increasing in terms of likelihood ratios $\frac{f_2}{f_1}$ and $\frac{g_2}{g_1}$, inequalities among the posterior probability distributions corresponding to the above four likelihood ratios are reversed. Now consider $V_x^2(\xi)$ and $V_y^2(\xi)$:

$$V_x^2(\xi) = c + V^2\left(\frac{1}{1 + \frac{1-\xi}{\xi}\frac{u}{p}}\right)(\xi p + (1-\xi)u)$$

$$+ V^2\left(\frac{1}{1 + \frac{1-\xi}{\xi}\frac{v}{q}}\right)(\xi q + (1-\xi)v) \quad ,$$

$$V_y^2(\xi) = c + V^2\left(\frac{1}{1 + \frac{1-\xi}{\xi}\frac{u'}{p'}}\right)(\xi p' + (1-\xi)u')$$

$$+ V^2\left(\frac{1}{1 + \frac{1-\xi}{\xi}\frac{v'}{q'}}\right)(\xi q' + (1-\xi)v') \quad .$$

Let $1_{z,\xi}(t)$ be the straight line joining $(T_z(\xi,0),\ V^2(T_z(\xi,0)))$ and $(T_z(\xi,1),\ V^2(T_z(\xi,1)))$, $Z = X$ or Y. By showing that

$$\frac{\frac{1}{1 + \frac{1-\xi}{\xi}\frac{u}{p}} - \xi}{\frac{1}{1 + \frac{1-\xi}{\xi}\frac{u}{p}} - \frac{1}{1 + \frac{1-\xi}{\xi}\frac{v}{q}}} = \xi q + (1-\xi)v \quad ,$$

and

$$\frac{\xi - \frac{1}{1 + \frac{1-\xi}{\xi}\frac{v}{q}}}{\frac{1}{1 + \frac{1-\xi}{\xi}\frac{u}{p}} - \frac{1}{1 + \frac{1-\xi}{\xi}\frac{v}{q}}} = \xi p + (1-\xi)u \quad ,$$

it is clear that

$$l_{x,\xi}(\xi) \;=\; V_x^2(\xi) - c \quad . \tag{4.32}$$

Similarly, we have

$$l_{y,\xi}(\xi) \;=\; V_y^2(\xi) - c \quad . \tag{4.33}$$

The relations (4.32) and (4.33) may be interpreted by the statement that the weighted average of the risks computed at the two possible posterior points corresponding to given prior ξ is exactly the linear interpolation at ξ between the two possible (posterior, risk) points corresponding to the two possible sampling outcomes.

In accordance with the terms appearing in Lemma IV.2, we now define:

$$a \triangleq \frac{1}{1 + \frac{1-\xi}{\xi}\frac{v}{q}} \quad ,$$

$$b \triangleq \frac{1}{1 + \frac{1+\xi}{\xi}\frac{u}{p}} \quad ,$$

$$c \triangleq \frac{1}{1 + \frac{1-\xi}{\xi}\frac{v'}{q'}} \quad ,$$

$$d \triangleq \frac{1}{1 + \frac{1-\xi}{\xi}\frac{u'}{p'}} \quad .$$

Since (4.31) holds, then we have $a \le c < d \le b$. $V^2(\xi)$ is concave so that we may consider $V^2(\xi)$ as h in Lemma IV.2, and also consider $\xi q + (1-\xi)v$ and $\xi q' + (1-\xi)v'$ respectively as α and β, so that $\alpha a + (1-\alpha)b$ and $\beta c + (1-\beta)d$ correspond to E(X) and E(Y) of Lemma IV.2, with $k = \xi$. Hence, in view of Lemma IV.2,

$$1_{x,\xi}(\xi) \leq 1_{y,\xi}(\xi) \quad . \tag{4.34}$$

In view of (4.32), (4.33) and (4.34), we have

$$V_x^2(\xi) \leq V_y^2(\xi) \quad , \quad \forall \xi \quad .$$

E. Source Expendability in Truncated Sequential Problems

In this section we shall discuss a concept of source expendability in the context of truncated sequential problems which is, of course, slightly different from the analogous concept in the context of unrestricted ones. We shall first define what we mean by "source M-expendability" in the context of a truncated sequential problem. The discussion in this section then goes on to show how the work in Bradt and Karlin (7) and the development in Section IV.D. are modified when truncated sequential problems are introduced. Further, we quantify by example the reasonably obvious proposition that sampling cost is less decisive, compared to wrong-decision loss, in the short run (N-truncation) than in the long run (non-truncation).

Let $V_N^2(\xi)$ be defined analogously to the definition of $V_N(\underline{\xi})$ in (3.5), with two sources considered now instead of one and two states of nature instead of an arbitrary finite number. Consider a sequence $\{V_N^2(\xi)\}_{N=0}^M$ of optimal Bayes risk functions for a given problem with two sources of information. Let $\{\Xi_c^N\}_{N=0}^M$ be the corresponding sequence of continuation regions. We also let $V_{z,N}^2$, $1 < N \leq M$, be the Bayes risk of the policy taking first observation Z, Z = X or Y, and then pursuing the optimal policy corresponding to $V^2_{N-1}(\xi)$ thereafter.

Definition IV.6

Source Y is said to be M-expendable if it is true, for all $1 < N \leq M$, that

$$V^2_{x,N}(\xi) \leq V^2_{y,N}(\xi) \quad , \quad \xi \in \Xi^N_c \quad . \tag{4.35}$$

In view of the above definition, even though the condition $R_x \leq R_y$ refers to the case of sampling precisely once, and 1-expendability refers to sampling either once or not at all, it is nevertheless clear that $R_x \leq R_y$ implies source Y is 1-expendable in the case of equal sampling cost. It is obvious that the reverse of the above statement is not generally true if we have a continuous random variable X (e.g., normal) and a discrete random variable Y (e.g., binomial).

It is obviously true, by Definition IV.2, (10) and Definition IV.6, that, if source Y is N-expendable, ∀N, then source Y is expendable. It is the main purpose of this section to point out that the reverse of the above proposition is not true; that is, we are able to construct a counter-example such that a particular source out of the two sources is expendable (in the long run) but it is not N-expendable (in the short run) for some N.

The counter-example involves the following binomial situation:

	X		Y	
	0	1	0	1
θ_1:	0.99	0.01	0.89	0.11
θ_2:	0.01	0.99	0.11	0.89

,

$L = 1$, $c_x = 0.1$, $c_y = 0.01$.

We have $\frac{\xi}{1-\xi} \le \frac{a}{b}$ iff $\xi \le \frac{a}{a+b}$, $a, b > 0$; then (cf. (4.26), (4.27) with $q = u = 0.01$, $q' = u' = 0.11$) we have

$$\left.\begin{aligned} R_x(\xi,0) &= \xi && \text{for } \xi \le 0.01 \\ &= 0.01 && \text{for } 0.01 < \xi \le 0.99 \\ &= 1 - \xi && \text{for } \xi > 0.99 \end{aligned}\right\} \tag{4.36}$$

and

$$\left.\begin{aligned} R_y(\xi,0) &= \xi && \text{for } \xi \le 0.11 \\ &= 0.11 && \text{for } 0.11 < \xi \le 0.89 \\ &= 1 - \xi && \text{for } \xi > 0.89 \end{aligned}\right\} \tag{4.37}$$

Also recall that

$$V^2_{x,1}(\xi) = c_x + R_x(\xi,0) \quad , \tag{4.38}$$

$$V^2_{y,1}(\xi) = c_y + R_y(\xi,0) \quad , \tag{4.39}$$

and, in analogy to (3.5) and (4.6),

$$V^2_1(\xi) = \min \{V^2_o(\xi),\ c_x + R_x(\xi,0),\ c_y + R_y(\xi,0)\} \quad . \tag{4.40}$$

In view of (4.36), (4.37), (4.38), (4.39) and (4.40), we then have

$$
\begin{aligned}
V_1^2(\xi) &= \xi && \text{for } 0 \le \xi < 0.11 \\
&= 0.11 && \text{for } 0.11 \le \xi < 0.89 \qquad (4.41)\\
&= 1 - \xi && \text{for } 0.89 \le \xi \le 1 \quad .
\end{aligned}
$$

It is clear (cf. (4.41)) that the continuation region Ξ_e^1 is (0.11, 0.89) and $V_{x,1}^2(\xi) \le V_{y,1}^2(\xi)$ for all ξ in this region; hence Y is 1-expendable.

We must still show that it is not true that Y is expendable. This will be done by constructing the optimal Bayes risk function $V^x(\xi)$ for using only X, and then showing that the policy using first y and thenceforth only X has a Bayes risk at $\xi = 0.22$ less than $V^x(0.11)$. Suppose then that Y is expendable; then we may show that the optimal Bayes risk $V^x(\xi)$ for using only X is given by

$$
\begin{aligned}
&\xi && \text{for } \xi \le 0.11 \quad , \\
&0.11 && \text{for } 0.11 < \xi \le 0.89 \quad , \qquad (4.42)\\
&1 - \xi && \text{for } \xi > 0.89 \quad ,
\end{aligned}
$$

which is exactly $V_1^2(\xi)$. To show the policy corresponding to the risk described by (4.42) is indeed the optimal Bayes policy using only X we must show (Theorem 10.2.1 of (3)) that

$$V^x(0.11) = 0.10 + \sum_{t=0}^{1} V^x(T_x(0.11,\ t))\ \bar{f}_{0.11}(t) \quad , \qquad (4.43)$$

and

$$V^x(0.89) = 0.10 + \sum_{t=0}^{1} V^x(T_x(0.89,\ t))\ \bar{f}_{0.89}(t) \quad . \qquad (4.44)$$

By virtue of (4.42), the computations involving R.H.S. terms of (4.43) and (4.44) reveal that the relations (4.43) and (4.44) are indeed true.

Let $V^{x|y}(\xi)$ be the Bayes risk of the policy taking first Y and then pursuing the optimal Bayes policy using only X. Then, at $\xi = 0.11$, we have

$$
\begin{aligned}
V^{x|y}(0.11) &= 0.01 + \sum_{t=0}^{1} V^{x}(T_y(0.11),\ t))\ \bar{g}_{0.11}(t) \\
&= 0.01 + V^{x}\ \frac{0.11^2}{0.11^2 + 0.89^2}\ (0.11^2 + 0.89^2) \\
&\quad + V^{x}\ \frac{0.11\ (0.89)}{0.11(0.89) + 0.89\ (0.11)}\ (0.11(0.89) + 0.89(0.11)) \quad . \\
&= 0.01 + V^{x}(0.01505)(0.8042) + V^{x}(0.5)(0.1958) \\
&= 0.01 + (0.01505)(0.8042) + (0.11)(0.1958) \\
&= 0.04364 \quad ,
\end{aligned}
$$

which is obviously less than 0.11, the value of $V^{x}(0.11)$. Therefore source Y is not expendable.

We can in fact show that X is expendable. Consider the policy using only Y with its corresponding continuation region $\frac{0.11^2}{0.11^2 + 0.89^2}$, $\frac{0.89^2}{0.11^2 + 0.89^2}$, and let $E_i^{\xi}(N)$ be ASN (average sample numver) at prior ξ where the expectation is w.r.t. the density g_i; $i = 1, 2$, and also let $\bar{E}^{\xi}(N)$ be ASN at prior ξ w.r.t. $\bar{g}_{\xi} = \xi g_1 + (1-\xi)g_2$. Then it is clear that

$$
\bar{E}^{\xi}(N) = \xi E_i^{\xi}(N) + (1-\xi)\ E_2^{\xi}(N) \quad . \tag{4.45}
$$

Ingwell (16) has given an algorithm for computing $E_i(N)$ for various ξ with respect to a given continuation region, which yields here

$$E_1^{0.5}(N) = 1 + 0.89\, E_1^{0.89}(N) + 0.11\, E_1^{0.11}(N)$$

$$E_1^{0.11}(N) = 1 + 0.89\, E_1^{0.5}(N)$$

$$E_1^{0.89}(N) = 1 + 0.11\, E_1^{0.5}(N) \quad .$$

Solving the above system of three linear equations, we have

$$\left.\begin{aligned} E_1^{0.5}(N) &= 2.4870 \\ E_1^{0.11}(N) &= 3.2134 \\ E_1^{0.89}(N) &= 1.2736 \quad . \end{aligned}\right\} \qquad (4.46)$$

Similarly, we have

$$\left.\begin{aligned} E_2^{0.5}(N) &= 2.4870 \\ E_2^{0.11}(N) &= 1.2736 \\ E_2^{0.89}(N) &= 3.2134 \quad . \end{aligned}\right\} \qquad (4.47)$$

In view of (4.45), (4.46) and (4.47), we then have

$$\left.\begin{aligned} \bar{E}^{0.5}(N) &= 2.4870 \\ \bar{E}^{0.11}(N) &= 1.4870 \\ \bar{E}^{0.89}(N) &= 1.4870 \quad . \end{aligned}\right\} \qquad (4.48)$$

Let ϕ be the policy using only y corresponding to the boundaries $\left(\frac{0.11^2}{0.11^2 + 0.89^2} , \frac{0.89^2}{0.11^2 + 0.89^2} \right)$. Then the risk of ϕ at $\xi = 0.5$ is given by

$$V^{\phi}(0.5) = c_y \bar{E}^{0.5}(N) + \alpha \quad ,$$

where α is the expected terminal decision loss. Hence

$$V^{\phi}(0.5) = 0.01(2.4870) + \frac{0.11^2}{0.11^2 + 0.89^2}$$
$$= 0.0399 \quad .$$

Since we have $V^Y(0.5) \leq V^{\phi}(0.5)$, where V^Y is the optimal Bayes risk using only Y, we then have

$$V^Y(0.5) \leq V^{\phi}(0.5) = 0.3999 < c_x = 0.10 \quad ,$$

and the concavity and symmetry of V^Y ensure that X is expendable.

We summarize the main conclusion drawn from the above example: there is a case such that, if we are allowed only a finite number of observations, then a particular inexpensive source e may be expendable in the sense of N-expendability. But using only source e for a sufficiently long period of time may nevertheless uniformly improve on the optimal Bayes policy restricted to not using e. That is, the sampling cost may be less decisive, compared to wrong-decision loss, in the short run than in the long run.

V. SYMMETRIC RISK FUNCTIONS

We may recall that several examples (cf. Example IV.1, Example IV.2) are set up for the sake of simplicity in terms of "symmetric" risk functions; the reason being not only that computation time is thereby reduced, but also that points to be made are more easily understood this way. We are thereby motivated to formalize this idea, and to give conditions for the symmetry of the optimal risk for several cases. It turns out that all of these cases essentially are special cases of the case of three states of nature and two sources of information, which we proceed now to analyze.

Let

$$\underline{\Xi} = \{\underline{\xi} = (\xi_1,\xi_2,\xi_3): \quad \xi_i \geq 0, \ \sum_{i=1}^{3} \xi_i = 1\}$$

be the class of prior probability distributions on a parameter triple $(\theta_1,\theta_2,\theta_3)$, where the i-th component of $\underline{\xi}$ is $P(\theta_i)$. We define $\pi(\underline{\xi})$ to be a permutation of $\underline{\xi}$.

<u>Definition V.1</u>

$V^2(\underline{\xi})$ is said to be symmetric on $\underline{\Xi}$ if

$$V^2(\underline{\xi}) = V^2(\pi(\underline{\xi})) \quad , \ \forall\pi, \ \ \underline{\xi} \in \underline{\Xi} \quad .$$

<u>Lemma V.1</u>

Let $\underline{\Xi}$ be the 3-dimensional Euclidean simplex and let $\{f_N: \ \underline{\Xi} \to R\}$ be a sequence of real-valued functions symmetric on $\underline{\Xi}$. If $f = \lim_{N\to\infty} f_N$, then f is also symmetric on $\underline{\Xi}$.

<u>Proof</u> Suppose that $f = \lim_{N\to\infty} f_N$ exists and is defined at $\underline{\xi}$, then, for any permutation π, we have

$$\begin{aligned} |f(\underline{\xi}) - f(\pi(\underline{\xi}))| &= |f(\underline{\xi}) - f_N(\underline{\xi}) - f(\pi(\underline{\xi})) + f_N(\pi(\underline{\xi})) \\ &\quad + f_N(\underline{\xi}) - f_N(\pi(\underline{\xi}))| \\ &\le |f(\underline{\xi}) - f_N(\underline{\xi})| + |f(\pi(\underline{\xi}) - f_N(\pi(\underline{\xi}))| \quad . \end{aligned}$$

The R.H.S. of the above inequality is arbitrary small for large enough N.

Let X and Y be independent real-valued random variables, and let f_i and g_i be densities of X and Y, respectively, under θ_i, $i = 1,2,3$, with respect to a suitable common σ-finite measure μ, where μ can be thought of either as the product measure of Lebesgue measures or the product measure of counting measures. The latter is singled out below for detailed attention, but modifications of the argument to cover the former are required.

<u>Definition V.2</u>

A set $\{f_1,f_2,f_3\}$ of densities on S with respect to counting measure is said to be pairwise symmetric if for any (i,j), $i,j = 1,2,3$, $i \neq j$, there exists a one-to-one transformation T^{ij} from S onto S such that

$$f_i(s) = f_j(T^{ij}(s)) \quad ,$$

$$f_j(s) = f_i(T^{ij}(s)) \quad ,$$

and

$$f_k(s) = f_k(T^{ij}(s)) \quad , \quad i \neq j \neq k \quad ; \quad i,j,k = 1,2,3$$

Note that in the case of Lebesgue measure, the additional requirement is needed that the Jacobians of the transformations T^{ij} all are unity.

Theorem V.1

Consider a Bayesian sequential decision problem with three states of nature, two sources of information and 0 - L decision loss. If both $\{f_1,f_2,f_3\}$ and $\{g_1,g_2,g_3\}$ are pairwise symmetric, then the optimal Bayes risk $V^2(\xi)$ is symmetric on Ξ .

Proof (For the case of counting measure.) The optimal Bayes risk evaluated at $\underline{\xi} = (\xi_1,\xi_2,\xi_3)$ is given by

$$V^2(\underline{\xi}) = \min \{V_o^2(\underline{\xi}),\ V_x^2(\underline{\xi}),\ V_y^2(\underline{\xi})\} \quad ,$$

where

$V_o^2(\underline{\xi})$ is now given by

$$V_o^2(\underline{\xi}) = \min \{(1-\xi_1)L,\ (1-\xi_2)L,\ (1-\xi_3)L\} \quad ,$$

and $V_x^2(\underline{\xi})$ and $V_y^2(\underline{\xi})$ have structures analogous to (4.7) and (4.8), respectively.

It is obvious that $V_o^2(\underline{\xi}) = V_o^2(\pi(\underline{\xi}))$, $\forall\pi$, $\forall\underline{\xi}$. Thus we need only show that $V_x^2(\underline{\xi}) = V_x^2(\pi(\underline{\xi}))$ and $V_y^2(\underline{\xi}) = V_y^2(\pi(\underline{\xi}))$, $\forall\pi$, $\forall\underline{\xi}$. In fact any permutation can be written as a product of pairwise permutations; therefore, it is sufficient to show that

$$V^2(\xi_1,\xi_2,\xi_3) \quad = \quad V^2(\xi_1,\xi_3,\xi_2) \quad , \quad \forall \underline{\xi} \quad . \tag{5.1}$$

The method of proof is outlined as follows:

1. Define $V^2_{N,x}(\underline{\xi})$, the otpimal N-truncated Bayes risk, conditional on sampling x first.
2. By induction show that $V^2_{N,x}(\underline{\xi})$ is symmetric.
3. By induction show that $V^2_N(\underline{\xi})$ is symmetric.
4. Apply Lemma V.1.

1. Let $V^2_{N,x}(\underline{\xi})$ be the Bayes risk of policy taking x first and thereafter following the optimal policy corresponding to $V^2_{N-1}(\underline{\xi})$, and similarly for $V^2_{N,y}(\underline{\xi})$. Consider:

$$V^2_{N+1,x}(\underline{\xi}) \quad = \quad c_x + \sum_S V^2_N \; (T_x(\underline{\xi},x))\bar{f}_{\underline{\xi}}(x) \qquad , \tag{5.2}$$

and

$$V^2_{N+1,x}(\pi(\underline{\xi})) \quad = \quad c_x + \sum_S V^2_N \; (T_x(\pi(\underline{\xi}),x)\bar{f}_{\pi(\underline{\xi})}(x) \qquad . \tag{5.3}$$

2. Now suppose that $V^2_N(\underline{\xi})$ is symmetric on $\underline{\Xi}$, and let π^* be pairwise permutation between components ξ_2 and ξ_3 of $\underline{\xi} = (\xi_1,\xi_2,\xi_3)$, that is $\pi^*(\xi_1,\xi_2,\xi_3) = (\xi_1,\xi_3,\xi_2)$. Since $\{f_1,f_2,f_3\}$ is pairwise symmetric, then there exists T^{23}_x, where the subscript "x" of T^{23}_x indicates that we are dealing with X, such that

$$f_1(x) \quad = \quad f_1(T^{23}_x(x)) \qquad ,$$

$$f_2(x) \quad = \quad f_3(T^{23}_x(x)) \qquad , \tag{5.4}$$

and

$$f_3(x) = f_2(T_x^{23}(x)) \quad , \quad x \in S \quad .$$

By definition, we have

$$\bar{f}_{\underline{\xi}}(x) = \xi_1 f_1(x) + \xi_2 f_2(x) + \xi_3 f_3(x) \quad , \tag{5.5}$$

$$\bar{f}_{\pi^*(\underline{\xi})}(x) = \xi_1 f_1(x) + \xi_3 f_2(x) + \xi_2 f_3(x) \quad . \tag{5.6}$$

In view of (5.4) and (5.6),

$$\begin{aligned} \bar{f}_{\pi^*(\underline{\xi})}(T_x^{23}(x)) &= \xi_1 f_1(T_x^{23}(x)) + \xi_3 f_2(T_x^{23}(x)) + \xi_2 f_3(T_x^{23}(x)) \\ &= \xi_1 f_1(x) + \xi_3 f_3(x) + \xi_2 f_2(x) \quad . \end{aligned} \tag{5.7}$$

The second equality comes from (5.6). It is equally true that

$$\bar{f}_{\pi^*(\underline{\xi})}(T_x^{23}(x)) = \bar{f}_{\underline{\xi}}(x) \quad , \quad x \in S \quad . \tag{5.8}$$

Consider:

$$T_x(\underline{\xi},x) = \frac{1}{\bar{f}_{\underline{\xi}}(x)} (\xi_1 f_1(x), \ \xi_2 f_2(x), \ \xi_3 f_3(x)) \quad , \tag{5.9}$$

and

$$T_x(\pi^*(\underline{\xi}),x) = \frac{1}{\bar{f}_{\pi^*(\underline{\xi})}(x)} (\xi_1 f_1(x), \ \xi_3 f_2(x), \ \xi_2 f_3(x)) \quad . \tag{5.10}$$

Hence,

$$\pi^*(T_x(\pi^*(\underline{\xi}),x) = \frac{1}{\bar{f}_{\pi^*(\underline{\xi})}(x)} (\xi_1 f_1(x), \ \xi_2 f_3(x), \ \xi_3 f_2(x)) \quad . \tag{5.11}$$

In view of (5.4), (5.8), (5.9) and (5.11), we have

$$\pi^*(T_x(\pi^*(\underline{\xi}),\ T_x^{23}(x))) = T(\underline{\xi},x) \quad , \quad x \in S \quad . \tag{5.12}$$

By symmetry of $V_N^2(\underline{\xi})$, it follows that

$$V_N^2(T_x(\pi^*(\underline{\xi}),\ x)) = V_N^2(\pi^*(T_x(\pi^*(\underline{\xi}),\ x))) \quad . \tag{5.13}$$

Taking $\pi = \pi^*$ in (5.3), then

$$\begin{aligned} V_{N+1,x}^2(\pi^*(\underline{\xi}) &= c_x + \sum_S V_N^2(T_x(\pi^*(\underline{\xi}),\ x))\ \bar{f}_{\pi^*(\underline{\xi})}(x) \\ &= c_x + \sum_S V_N^2(\pi^*(T_x(\pi^*(\underline{\xi}),\ x)))\ \bar{f}_{\pi^*(\underline{\xi})}(x) \\ &= c_x + \sum_S V_N^2(\pi^*(T_x(\pi^*(\underline{\xi}),\ T_x^{23}(x))))\ \bar{f}_{\pi^*(\underline{\xi})}(T_x^{23}(x)) \\ &= c_x + \sum_S V_N^2(T_x(\underline{\xi},x))\ \bar{f}_{\underline{\xi}}(x) \\ &= V_{N+1,x}^2(\underline{\xi}) \quad . \end{aligned}$$

The first equality of the above is by definition, the second equality by symmetry of $V_N^2(\underline{\xi})$, the third equality by applying T_x^{23} on S, the fourth equality by (5.8), (5.12) and (5.13), and, finally, the fifth equality also by definition. By an analogous argument, we are able to show that $V_{N+1,x}^2(\underline{\xi}) = V_{N+1,x}^2(\pi(\underline{\xi}))$, $\forall\pi$, $\forall\underline{\xi}$.

3. Similarly $V_{N+1,y}^2(\underline{\xi}) = V_{N+1,y}^2(\pi(\underline{\xi}))$, $\forall\pi$, $\forall\underline{\xi}$ so that $V_{N+1}^2(\underline{\xi})$ is symmetric on $\underline{\Xi}$.

4. Since it is true that

$$V_N^2(\underline{\xi}) = \min \{V_o^2(\underline{\xi}),\ V_{N,x}^2(\underline{\xi}),\ V_{N,y}^2(\underline{\xi})\} \quad , \quad \forall N \quad ,$$

we have that $V_N^2(\underline{\xi})$ is bounded below and non-increasing, and hence that $V^2(\underline{\xi}) = \lim_{N\to\infty} V_N^2(\underline{\xi})$ exists. Therefore $V^2(\underline{\xi})$ is symmetric on $\underline{\Xi}$, since $V_N^2(\underline{\xi})$ is symmetric on $\underline{\Xi}$, by Lemma V.1.

As an illustration, consider the following trinomial situation where X and Y have densities given by

	X			Y		
	0	1	2	0	1	2
θ_1:	a	a	b	c	c	d
θ_2:	a	b	a	c	d	c
θ_3:	b	a	a	d	c	c

It is clear that both $\{f_1,f_2,f_3\}$ and $\{g_1,g_2,g_3\}$ of the above table are pairwise symmetric. The above illustration gives an indication how one can also construct densities for X and Y with respect to Lebesgue measure that lead to symmetry of V^2 in a similar fashion; that is, one is able to construct tri-modal densities with respect to Lebesgue measure such that one of these modes represents "b" and the other two represent "a" as illustrated in the above table, and in such a way that the mode "b" occur at symmetrically position points of sample space for the three different states of nature. It is also true that there are bivariate random variables with densities with respect to product of Lebesgue measure, leading to symmetry. As an example, consider bivariate normal random variable with densities under states of nature given by

θ_1: $N\left(\binom{1}{0}, \sigma^2 I\right)$

θ_2: $N\left(\binom{-1}{0}, \sigma^2 I\right)$

θ_3: $N\left(\binom{0}{\sqrt{3}}, \sigma^2 I\right)$, $\sigma^2 > 0$, I is identity matrix ,

and consider

$$T^{13}(x_1,x_2) = \frac{x_1 + \sqrt{3}x_2 - 1}{2}, \frac{\sqrt{3}x_1 - x_2 + \sqrt{3}}{2} ,$$

which we can verify that

$$f_1(T^{13}(x_1,x_2)) = f_3(x_1,x_2) ,$$

$$f_3(T^{13}(x_1,x_2)) = f_1(x_1,x_2) ,$$

and

$$f_2(T^{13}(x_1,x_2)) = f_2(x_1,x_2) .$$

A final remark concerning Theorem V.1 is that the result holds as well for the case of one source of information, which is obtained as a special case by assigning prohibitive cost to one of the two sources, and also for the case of two states of nature, which is obtained as a special case by assigning a prior probability distribution degenerate with respect to one of the three states of nature.

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VII. ACKNOWLEDGMENTS

I sincerely wish to thank my major professor Dr. H. T. David for his endless support throughout this dissertation. This work would never have been finished without his availability, academic knowledge, motivation and special care. I honestly say he is one of the concrete foundations for my future life.

Research in connection with this dissertation was partially supported by U.S. Air Force Office of Scientific Research grant AFOSR-78-3518.

Thanks are also given to the Thai Government in its financial support and permission.

I am grateful to my parents, my sisters and my brothers for their love and patient waiting.